Supra-convergences of Shortley-Weller method for Poisson Equation

Gangjoon Yoon*and Chohong Min†

March 21, 2014

Abstract

The Shortley-Weller method is a basic finite difference method for solving the Poisson equation with Dirichlet boundary condition. The second order convergence of its solution has been long known but it is rather recent to pay attention to its gradient. Especially in its application to fluid flows, the gradient plays a physical role rather than the solution itself. In this article, we first review the proof that the convergence order of its numerical solution is the second order: though consistency error is first order accurate at some locations, the convergence order is globally second order. We call this increase of the order of accuracy, supra-convergence. We then discuss a discrete divergence theorem for the Shortley-Weller method and prove that the gradient of the solution is second order accurate in general domains. Usually, the gradient of a second order solution is only first order accurate, but the gradient of the Shortley-Weller solution is second order accurate, which is another supra-convergence.

1 Introduction

The Poisson equation $-\Delta u = f$ is of primal importance in many physical problems, especially in fluid flows with incompressible condition. The pressure variable in the flows satisfies the Poisson equation with Dirichlet boundary condition at free surface and Neumann boundary condition at solid surface [15]. One important aspect of a Poisson solver is the ability to deal with both boundary conditions, and the other is the accuracy of the gradient of the solution, for pressure gradient is a physical variable in fluid flows rather than the pressure itself.

Except for some particular cases, the exact solution of the Poisson equation is unknown and needs to be approximated. Finite difference methods, finite element methods, and boundary integral methods are main tools for the approximation. In general, finite difference methods have advantages in grid generation and may have difficulties in treating irregular boundary. The three main tools have their own pros and cons, and the choice among them depends on the given problem. In this article, we confine our discussion to finite difference methods.

The Shortley-Weller method [18] is a basic finite difference method for solving the Poisson equation with Dirichlet boundary condition. It is a simple dimension-by-dimension approach that works in any dimensions. The method results in a non-symmetric linear system whose matrix is an M-matrix. It was proved in [18, 3] that the numerical solution is second order accuarate. The gradient of the solution was numerically observed to be second order [13], but the observation has not been proved yet.

The work of Gibou et al. [5] is a simple modification of the Shortley-Weller method. The modification results in symmetric liner system, which can be solved more efficiently than the non-symmetric one. Its numerical solution is still second order accurate but the accuracy of the gradient drops to first order.

*Institute of Mathematical Sciences, Ewha Womans University, Seoul, Korea 120-750

†Mathematics Department, Ewha Womans University, Seoul, Korea 120-750, corresponding author(chohong@ewha.ac.kr)

The work of Purvis [16, 14] solves the Poisson equation with the homogeneous Neumann boundary condition. It is a finite volume approach that reads as a standard five-point finite difference method for the Poisson equation with a weight function which is the characteristic function of the domain. The method results in symmetric linear system. It was numerically reported that both of the numerical solution and its gradient are second order accurate. In the case of non-homogeneous Neumann boundary condition, it was observed that the solution keeps the second order accuracy, but the gradient is only first order accurate [13]. These observations have not been proved yet up to our best search.

We have listed three basic finite difference methods for the Poisson equation with Dirichlet boundary condition or with Neumann boundary condition. Contrary to their great importance, their convergence properties of the finite difference methods have just been taken for granted from numerical tests, not from concrete proof. Convergence analysis for solution and its gradient has been well studied in finite element methods [9, 7, 8].

The main theme of this article is the convergence analysis for the Shortley-Weller method that solves the Poisson equation with Dirichlet boundary condition. The second order convergence of its solution has been well known [18, 3, 1]. Matsunaga-Yamamoto [12] improved the result by showing the third order accuracy near the boundary. Similar results have been obtained for nonsmooth Dirichlet problem [2] and convection-diffusion problem [4]. It is rather recent to pay attention to its gradient. The gradient of the solution was numerically observed to be second order accurate in general domains [13], but the mathematical proof for the observation has not been reported yet. The second order convergence was proved in rectangular domain [11], and the order of one and a half was proved in polygonal domains [10]. In this article, we prove the second order convergence in general domains.

We first briefly review the proof that the convergence order of its numerical solution is the second order. Though consistency error is first order accurate at some locations. The convergence order is globally second order. We call this increase of the order of accuracy, supra-convergence. We then discuss a discrete divergence theorem for the Shortley-Weller method and prove that the gradient of the solution is second order accurate. Usually, the gradient of a second order solution is only first order accurate, but the gradient of the Shortley-Weller solution is second order accurate, which is another supra-convergence.

2 Discretization Setting

In this section, we define discretizations of domain and differential operators for solving the Poisson problem

$$\begin{cases} -\Delta u &= f \quad \text{in} \quad \Omega \\ \quad u &= g \quad \text{on} \quad \Gamma, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is an open and bounded domain with smooth boundary Γ. Consider a uniform grid with step size h, i.e. $h\mathbb{Z}^2$. By Ω_h we denote the set of grid nodes belonging to Ω, and Γ_h denotes the set of intersection points between Γ and grid lines, i.e. $\Omega_h = \Omega \cap \left(h\mathbb{Z}^2\right)$ and $\Gamma_h = \Gamma \cap \{(h\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times h\mathbb{Z})\}$. As illustrated in Figure 1, a grid node $(x_i, y_i) \in \Omega_h$ has four neighboring nodes in $\Omega_h \cup \Gamma_h$, namely $(x_{i\pm1}, y_j)$ and $(x_i, y_{j\pm1})$ in $\Omega_h \cup \Gamma_h$. Let $h_{i+\frac{1}{2},j}$ denote the distance from (x_i, y_j) to its neighbor (x_{i+1}, y_j), and other distances $h_{i-\frac{1}{2},j}$, $h_{i,j\pm\frac{1}{2}}$ are defined in the same fashion.

Now we move on to the discretization of differential operators. Given a discrete function $u : \Omega_h \cup \Gamma_h \to \mathbb{R}$, its derivative in x-direction is naturally calculated as

$$(D_h^x u)_{i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{h_{i+\frac{1}{2},j}},$$

and it is defined at the middle point $\left(\frac{x_i+x_{i+1}}{2}, y_j\right)$ whenever $(x_i, y_j) \in \Omega_h$ or $(x_{i+1}, y_j) \in \Omega_h$. The

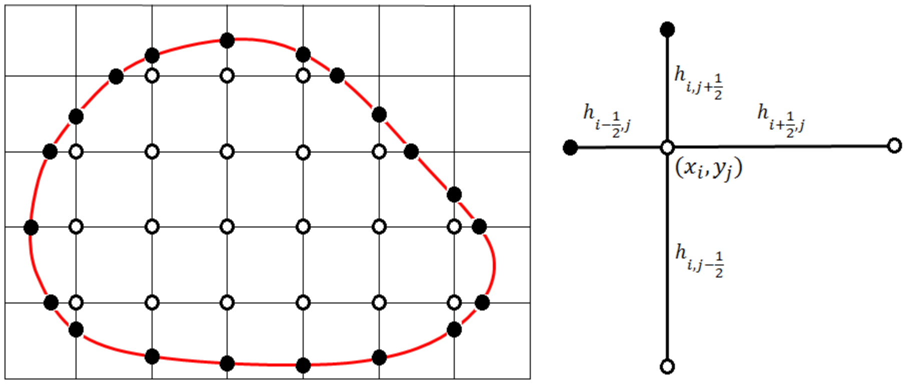

Figure 1: Grid nodes in Ω_h are marked by $\circ$ and nodes in Γ_h by $\bullet$. A grid node $(x_i, y_j) \in \Omega_h$ has four neighboring nodes in $\Omega_h \cup \Gamma_h$.

second derivative $D_h^{xx}u : \Omega_h \to \mathbb{R}$ is calculated as

$$(D_h^{xx}u)_{ij} = \frac{(D_h^x u)_{i+\frac{1}{2},j} - (D_h^x u)_{i-\frac{1}{2},j}}{\frac{h_{i+\frac{1}{2},j}+h_{i-\frac{1}{2},j}}{2}}, \tag{1}$$

for each $(x_i, y_j) \in \Omega_h$. $(D_h^y u)_{ij+\frac{1}{2}}$ and $(D_h^{yy}u)_{ij}$ are similarly defined. The Shortley-Weller discretization of the Laplace operator is then defined as $\Delta_h = D_h^{xx} + D_h^{yy}$. In specific, given a function $u : \Omega_h \cup \Gamma_h \to \mathbb{R}$, its Laplacian $\Delta_h u : \Omega_h \to \mathbb{R}$ is defined as

$$(\Delta_h u)_{ij} = \left(\frac{u_{i+1,j} - u_{ij}}{h_{i+\frac{1}{2},j}} - \frac{u_{ij} - u_{i-1,j}}{h_{i-\frac{1}{2},j}} \right) \frac{2}{h_{i+\frac{1}{2},j} + h_{i-\frac{1}{2},j}} + \left(\frac{u_{i,j+1} - u_{ij}}{h_{i,j+\frac{1}{2}}} - \frac{u_{ij} - u_{i,j-1}}{h_{i,j-\frac{1}{2}}} \right) \frac{2}{h_{i,j+\frac{1}{2}} + h_{i,j-\frac{1}{2}}} .$$

3 Supra-convergence of Solution

Let $u : \Omega \to \mathbb{R}$ be the continuous solution of the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega \\ u(x) = g(x), & x \in \Gamma, \end{cases} \tag{2}$$

and let $u_h : \Omega_h \cup \Gamma_h \to \mathbb{R}$ be the solution of the discrete equation

$$\begin{cases} -\Delta_h u_h(P) = f(P), & P \in \Omega_h \\ u_h(P) = g(P), & P \in \Gamma_h. \end{cases} \tag{3}$$

In this section, we briefly review the proof in [3, 18] that the discrete solution approximates the continuous solution with the second order accuracy. Though the consistency order of the discretization ranges from the first to the second, its convergence order is the second order everywhere. In some regions, that we will specify shortly later, convergence order is one more than consistency order. we call such gain of order *supra-convergence*, or super-convergence. Most of lemmas and theorems in this section will be just stated without proofs, which we refer to [3] for details, for our main theme of this work is to introduce the supra-convergence of the gradient of the discrete solution.

Definition $\Omega_h^* \subset \Omega_h$ denotes the set of grid nodes adjacent to Γ_h, and $\Omega_h^\circ = \Omega_h \setminus \Omega_h^*$.

Lemma 3.1 *(Consistency error) A simple Taylor series expansion shows that*

$$(-\Delta_h (u - u_h))(P) = \begin{cases} O(h^2), & P \in \Omega_h^\circ \\ O(h), & P \in \Omega_h^*. \end{cases} \tag{4}$$

The discrete equation (3) for each $P \in \Omega_h$ form a non-symmetric linear system whose matrix is an M-matrix [17]. An important property of an M-matrix is that its inverse is non-negative in every entry, from which the following discrete maximum principle follows.

Lemma 3.2 *(Discrete maximum principle) If $-\Delta_h u \geq 0$ then the minimum value of u should be achieved on Γ_h. Similarly, if $-\Delta_h u \leq 0$ then the maximum value of u should be achieved on Γ_h.*

Definition (Discrete Green's function) For each $Q \in \Omega_h$, we define the function $G_h(P,Q)$, $P \in \Omega_h \cup \Gamma_h$ as the solution of the discrete problem

$$\begin{cases} -\Delta_h u_h(P) = \begin{cases} 0, & P \neq Q \\ \frac{1}{h^2}, & P = Q \end{cases}, & P \in \Omega_h \\ u_h(P) = 0, & P \in \Gamma_h. \end{cases} \tag{5}$$

Since $-\Delta_h u_h \geq 0$, the minimum should be achieved on Γ_h, and therefore $G_h(P,Q) \geq 0$ for any $P \in \Omega_h \cup \Gamma_h$. The Green functions $G(\cdot, Q)$ generate all functions u_h on $\Omega_h \cap \Gamma_h$ which is zero on Γ_h as follows :

Lemma 3.3 *(Expansion by Green's function) Let u_h be a function defined on $\Omega_h \cap \Gamma_h$ with $u_h \equiv 0$ on Γ_h, then we have the representation for u_h,*

$$u_h(P) = \sum_{Q \in \Omega_h} (-\Delta_h u_h(Q)) G_h(P,Q) h^2, \quad \textit{for all } P \in \Omega_h \cup \Gamma_h. \tag{6}$$

Using the maximum principle in comparison between u_h and a continuous function U satisfying $-\Delta_h U = 1$ in Ω_h and $U = 0$ on Γ_h, the following estimates are obtained.

Lemma 3.4 *(Bounds for Green's function)*

(i) There is a constant C independent of h such that

$$\sum_{Q \in \Omega_h} G_h(P,Q) h^2 \leq C, \quad \textit{for all } P \in \Omega_h \cup \Gamma_h.$$

(ii)

$$\sum_{Q \in \Omega_h^*} G_h(P,Q) \leq 1, \quad \textit{for all } P \in \Omega_h \cup \Gamma_h.$$

Now, combining the lemmas leads to the proof for the supra-convergence of solution.

Theorem 3.5 *(Supra-convergence of solution) Let u be a continuous solution to the problem (2) and u_h a discrete solution to the problem (3). For any $P \in \Omega_h \cup \Gamma_h$, we have that $u(P) - u_h(P) = 0$ if $P \in \Gamma_h$ and*

$$u(P) - u_h(P) = O\left(h^2\right), \ \textit{for all } P \in \Omega_h.$$

Proof Since $u - u_h = 0$ on Γ_h, the summation formula holds for all $P \in \Omega_h \cup \Gamma_h$, and we have

$$\begin{aligned} (u-u_h)(P) &= \sum_{Q \in \Omega_h} (-\Delta_h (u-u_h)(Q)) G_h(P,Q) h^2 \\ &= \sum_{Q \in \Omega_h^*} (-\Delta_h (u-u_h)(Q)) G_h(P,Q) h^2 + \sum_{Q \in \Omega_h^\circ} (-\Delta_h (u-u_h)(Q)) G_h(P,Q) h^2 \\ &= O(h) \left(\sum_{Q \in \Omega_h^*} G_h(P,Q) \right) h^2 + O\left(h^2\right) \left(\sum_{Q \in \Omega_h^\circ} G_h(P,Q) h^2 \right) \\ &= O\left(h^3\right) + O\left(h^2\right) = O\left(h^2\right), \end{aligned}$$

and this proves the theorem. □

4 Supra-Convergence of Gradient

The gradient of a second order accurate solution is usually first order accurate, but in some methods for elliptic problems the gradient keeps the second order accuracy, which we also call this gain of order supra-convergence. Both solutions of the Gibou's method [5] and the Shortley-Weller's method [18] are second order accurate, but the solution gradient of Shortley-Weller's preserve the second order accuracy while that of Gibou's drops to the first order. These were observed in thorough numerical tests [13]. In this section, we analyze and prove the supra-convergence on the gradient of the Shortley-Weller's.

A classical reference [19] shows the supra-convergence in rectangular domains whose boundaries are aligned at grid lines. Though its presentation was complicated with Fourier analysis, its main idea is to simply take a discrete divergence theorem on error $e = u - u_h$,

$$\sqrt{\int_{\Omega_h} |\nabla_h e|^2 d\Omega_h} = \sqrt{\int_{\Omega_h} e \cdot (-\Delta_h e)\, d\Omega_h} = \sqrt{O\left(h^2\right) O\left(h^2\right) \int_{\Omega_h} d\Omega_h} = O\left(h^2\right).$$

In general irregular domains, the boundary of domain is not aligned with grid lines, which makes a residue in the application of a discrete divergence theorem. In our review paper [20], we pointed out that the discrete divergence theorem is not valid any more in irregular domains, and suggested that a new discrete divergence theorem suiting with the Shortley-Weller discretization may lead to the proof of the supra-convergence of gradient.

In the beginning of this section, we define discrete integrals in irregular domains. The definition copes with the definition of the discrete Laplacian in Section 2, and we can derive a discrete divergence theorem to identify the residue term. Then we extend the estimates of Green's function in Section 3 for treating the residue, and finally proceed to the proof of the supra-convergence on gradient.

4.1 Discrete Divergence Theorem

A grid node (x_i, y_j) in Ω_h has four neighboring nodes $(x_{i\pm1}, y_j)$ and $(x_i, y_{j\pm1})$ in $\Omega_h \cup \Gamma_h$, and accordingly we define its control volume such that its border line is up to the middle of the node and its neighbor in each four direction,

$$C_{ij} := \left[\frac{x_i + x_{i-1}}{2}, \frac{x_i + x_{i+1}}{2}\right] \times \left[\frac{y_j + y_{j-1}}{2}, \frac{y_j + y_{j+1}}{2}\right].$$

The control volume is a rectangle of size $\frac{1}{2}\left(h_{i+\frac{1}{2},j} + h_{i-\frac{1}{2},j}\right) \times \frac{1}{2}\left(h_{i,j+\frac{1}{2}} + h_{i,j-\frac{1}{2}}\right)$. The control volumes of two neighboring nodes are adjacent along the border at their middle point. In overall, their union $C_h = \cup_{(x_i,y_j)\in\Omega_h} C_{ij}$ seamlessly fills up the domain Ω inside, but some margins between C and Ω appear near the boundary Γ, as depicted in Figure 2.

The L_2 inner-product between two discrete functions $u, v : \Omega_h \to \mathbb{R}$ is defined as the multiplication of their values and the area of the control volume for each grid node,

$$\int_{\Omega_h} u \cdot v \, d\Omega_h := \sum_{(x_i,y_j)\in\Omega_h} u_{ij} v_{ij} \frac{h_{i+\frac{1}{2},j} + h_{i-\frac{1}{2},j}}{2} \frac{h_{i,j+\frac{1}{2}} + h_{i,j-\frac{1}{2}}}{2}. \tag{7}$$

Now let us proceed to the definition of the H_1 semi-inner-product $\int_{\Omega_h} \nabla_h u \cdot \nabla_h v \, d\Omega_h$. Consider two discrete functions $u, v : \Omega_h \cup \Gamma_h \to \mathbb{R}$. For each grid node $(x_i, y_j) \in \Omega_h$, we have two one-sided approximations, $\frac{u_{i+1,j} - u_{ij}}{h_{i+\frac{1}{2},j}}$ and $\frac{u_{ij} - u_{i-1j}}{h_{i-\frac{1}{2},j}}$ for $\frac{\partial u}{\partial x}(x_i, y_j)$. In calculating $\int_{C_{ij}} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dxdy$, we split the control volume into the region $\left[x_{i-\frac{1}{2}}, x_i\right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right]$ left to the grid node and the region $\left[x_i, x_{i+\frac{1}{2}}\right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right]$ right, and match the left region to the approximation from the left and the right region to the one from the right,

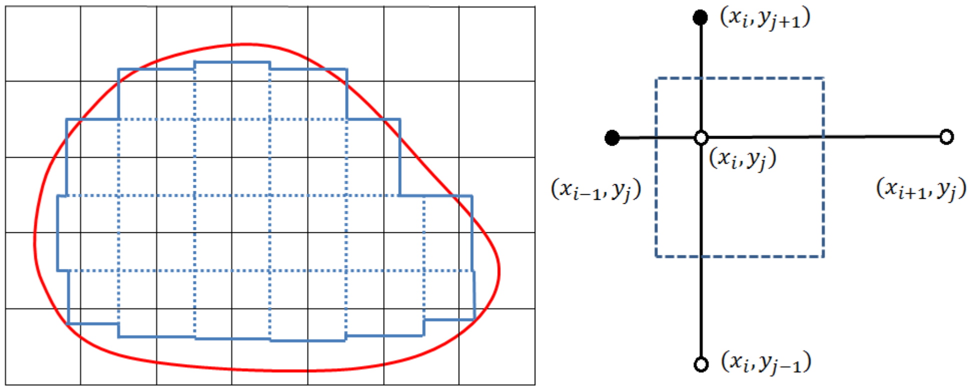

Figure 2: Each control volume C_{ij} is depicted in dotted lines. A union of control volumes $C_h = \cup_{(x_i,y_j)} C_{ij}$ fills up seamlessly inside the domain Ω with some margin near Γ. The boundary of the union, ∂C_h, is depicted in solid straight lines.

$$\int\limits_{\Omega_h} (D_h^x u)\,(D_h^x v)\,d\Omega := \sum_{(x_i,y_j)\in\Omega_h} \left(\begin{array}{cc} & \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{i+1,j}-v_{ij}}{h_{i+\frac{1}{2},j}} \frac{h_{i+\frac{1}{2},j}}{2} \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \\ + & \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{ij}-v_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{h_{i-\frac{1}{2},j}}{2} \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \end{array} \right). \tag{8}$$

In the same fashion, $\int_{\Omega_h} (D_h^y u)\,(D_h^y v)\,d\Omega$ is defined. Our treatment of u_{ij}, $(D_h^x u)_{i+\frac{1}{2}j}$ and $(D_h^y u)_{ij+\frac{1}{2}}$ is similar to the Marker-and-Cell discretization in staggered grids [6]. For each control volume C_{ij}, we have u_{ij} at the grid node inside, $(D_h^x u)_{i+\frac{1}{2},j}$ and $(D_h^x u)_{i-\frac{1}{2},j}$ at its left and right borders, respectively, and $(D_h^y u)_{ij\pm\frac{1}{2}}$ at its top and bottom borders.

Before we state and prove a discrete divergence theorem, we need to deal with another discretization, how to approximate line integral $\int_\Gamma \frac{\partial u}{\partial x} v\,(n\cdot e_1)\,d\Gamma$, one of the two components in $\int_\Gamma (\nabla u\cdot n)\,v\,d\Gamma$. As depicted in Figure 2, the margin between Ω and $C_h = \cup_{(x_i,y_j)} C_{ij}$ is present only near Γ. Hence we approximate the support Γ by ∂C_h. For each control volume C_{ij}, the integral over its boundary is defined as the sampled value times the length summed over its four sides,

$$\int\limits_{\partial C_{ij}} (D_h^x u)\,v\,(n\cdot e_1)\,d\Gamma := \left(\begin{array}{cc} + & \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{i+1,j}+v_{ij}}{2} \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \\ - & \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{ij}+v_{i-1j}}{2} \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \end{array} \right).$$

Note that $n\cdot e_1 = 0$ at the top and bottom sides, so the above sum has only two terms from the left and right borders. Summing up the oriented line integrals over all the control volumes, we have

$$\int\limits_{\partial C_h} (D_h^x u)\,v\,(n\cdot e_1)\,d\Gamma = \sum_{(x_i,y_j)\in\Omega_h} \left(\begin{array}{cc} + & \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{i+1,j}+v_{ij}}{2} \\ - & \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{ij}+v_{i-1j}}{2} \end{array} \right) \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2}. \tag{9}$$

Inside the domain, an edge appears twice in the summation with different signs and the two terms cancel out each other. Hence the support of the summation actually runs only over ∂C_h. The terms with the dotted lines in Figure 2 are all canceled out.

Theorem 4.1 *(Discrete integration-by-parts) For any $u, v : \Omega_h \cup \Gamma_h \to \mathbb{R}$,*

$$\int\limits_{\Omega_h} (D_h^{xx} u)\,v\,d\Omega_h + \int\limits_{\Omega_h} (D_h^x u)\,(D_h^x v)\,d\Omega_h = \int\limits_{\partial C_h} (D_h^x u)\,v\,(n\cdot e_1)\,d\Gamma.$$

Proof From the definitions (1) and (7),

$$
\begin{aligned}
\int_{\Omega_h} (D_h^{xx}u)\, v\, d\Omega_h &= \sum_{(x_i,y_j)\in\Omega_h} (D_h^{xx}u)_{ij}\, v_{ij} \frac{h_{i+\frac{1}{2},j}+h_{i-\frac{1}{2},j}}{2} \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h} \left(\frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} - \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \right) v_{ij} \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h} \left(\frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{ij}+v_{ij}}{2} - \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{ij}+v_{ij}}{2} \right) \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2}
\end{aligned}
$$

From the definition (8),

$$
\begin{aligned}
\int_{\Omega_h} (D_h^{x}u)\,(D_h^{x}v)\, d\Omega_h &= \sum_{(x_i,y_j)\in\Omega_h} \left(\begin{array}{cc} & \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{i+1,j}-v_{ij}}{h_{i+\frac{1}{2},j}} \frac{h_{i+\frac{1}{2},j}}{2} \\ + & \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{ij}-v_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{h_{i-\frac{1}{2},j}}{2} \end{array} \right) \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h} \left(\begin{array}{cc} & \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{i+1,j}-v_{ij}}{2} \\ + & \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{ij}-v_{i-1j}}{2} \end{array} \right) \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2}
\end{aligned}
$$

and then the sum of two integrals is calculated as

$$
\begin{aligned}
\int_{\Omega_h} (D_h^{xx}u)\, v\, d\Omega_h + \int_{\Omega_h} (D_h^{x}u)\,(D_h^{x}v)\, d\Omega_h &= \sum_{(x_i,y_j)\in\Omega_h} \left(\begin{array}{cc} + & \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \frac{v_{i+1,j}+v_{ij}}{2} \\ - & \frac{u_{ij}-u_{i-1j}}{h_{i-\frac{1}{2},j}} \frac{v_{i-1j}+v_{ij}}{2} \end{array} \right) \frac{h_{i,j+\frac{1}{2}}+h_{ij-\frac{1}{2}}}{2} \\
&= \int_{\partial C_h} (D_h^{x}u)\, v\, (n\cdot e_1)\, d\Gamma.
\end{aligned}
$$

This shows the discrete version of integration-by-parts. □

Repeating the above process in y-direction, we obtain the discrete divergence theorem.

Corollary 4.2 *(Discrete divergence theorem) For any $u, v : \Omega_h \cup \Gamma_h \to \mathbb{R}$,*

$$
\int_{\Omega_h} (\Delta_h u)\, v\, d\Omega_h + \int_{\Omega_h} (\nabla_h u)\,(\nabla_h v)\, d\Omega_h = \int_{\partial C_h} (\nabla_h u \cdot n)\, v\, d\Gamma. \tag{10}
$$

4.2 Approximation of Gradient

Given the discrete solution u_h approximating the continuous solution u, the derivatives of u are then approximated by the finite differences of u_h in the staggered grid nodes. For example, using two neighboring grid nodes (x_i, y_j), $(x_{i+1}, y_j) \in \Omega_h \cup \Gamma_h$, we have

$$
\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, y_j \right) \simeq \frac{(u_h)_{i+1,j} - (u_h)_{ij}}{h_{i+\frac{1}{2},j}}.
$$

A standard result for central finite differences gives

$$
\frac{\partial u}{\partial x}\left(\frac{x_i + x_{i+1}}{2}, y_j \right) = \frac{u\,(x_{i+1}, y_j) - u\,(x_i, y_j)}{h_{i+\frac{1}{2},j}} + O\left(h^2_{i+\frac{1}{2},j} \right).
$$

Since our goal in this work is to show that the gradient approximation is second order accurate and since $h_{i+\frac{1}{2},j} \leq h$, it is enough that the following approximation is second order accurate,

$$\frac{u\left(x_{i+1}, y_j\right) - u\left(x_i, y_j\right)}{h_{i+\frac{1}{2},j}} \simeq \frac{\left(u_h\right)_{i+1,j} - \left(u_h\right)_{ij}}{h_{i+\frac{1}{2},j}}$$

The error of the approximation is simply $\left(D_h^x e_h\right)_{i+\frac{1}{2},j}$, where $e_h = u - u_h$. Hence, hereafter we focus on measuring the size of $D_h^x e_h$ and $D_h^y e_h$, or the size of $\left|\nabla_h e_h\right|$. Applying the discrete divergence theorem on e_h, we have

$$\int_{\Omega_h} \left|\nabla_h e_h\right|^2 d\Omega_h = \int_{\Omega_h} \left(-\Delta_h e_h\right) e_h \, d\Omega_h + \int_{\partial C_h} \left(\nabla_h e_h \cdot n\right) e_h \, d\Gamma.$$

The following lemma estimates the first integral in the right hand side of the equation above.

Lemma 4.3 $\int_{\Omega_h} \left(-\Delta_h e_h\right) e_h \, d\Omega_h = O\left(h^4\right)$

Proof Note that $-\Delta_h e_h = \Delta_h u_h - \Delta_h u = \Delta u - \Delta_h u$ is nothing but the consistency error. Using Lemma 3.1 and Theorem 3.5, we have

$$\begin{aligned}
\int_{\Omega_h} \left(-\Delta_h e_h\right) e_h \, d\Omega_h &= \sum_{(x_i,y_j)\in\Omega_h} \left(-\Delta_h e_h\right)_{ij} \left(e_h\right)_{ij} \frac{h_{i-\frac{1}{2},j} + h_{i+\frac{1}{2},j}}{2} \frac{h_{i,j-\frac{1}{2}} + h_{i,j+\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h^\circ} \left(-\Delta_h e_h\right)_{ij} \left(e_h\right)_{ij} \frac{h_{i-\frac{1}{2},j} + h_{i+\frac{1}{2},j}}{2} \frac{h_{i,j-\frac{1}{2}} + h_{i,j+\frac{1}{2}}}{2} \\
&\quad + \sum_{(x_i,y_j)\in\Omega_h^*} \left(-\Delta_h e_h\right)_{ij} \left(e_h\right)_{ij} \frac{h_{i-\frac{1}{2},j} + h_{i+\frac{1}{2},j}}{2} \frac{h_{i,j-\frac{1}{2}} + h_{i,j+\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h^\circ} O\left(h^{2+2}\right) O\left(h^2\right) + \sum_{(x_i,y_j)\in\Omega_h^*} O\left(h^{1+2}\right) O\left(h^2\right) \\
&= O\left(h^6\right) O\left(h^{-2}\right) + O\left(h^5\right) O\left(h^{-1}\right) = O\left(h^4\right).
\end{aligned}$$

Here we used the fact that since Ω is a domain in two dimensions, the number of grid nodes in Ω_h or in Ω_h° grows as $O\left(h^{-2}\right)$, and since the boundary Γ is one dimensional and the grid nodes in Ω_h^* are present only near Γ, the number of grid nodes in Ω_h^* grows as $O\left(h^{-1}\right)$. $\square$

4.3 Convergence of Gradient

Now let us proceed to the estimation of the second integral $\int_{\partial C_h} \left(\nabla_h e_h \cdot n\right) e_h \, d\Gamma$. The estimate in Theorem 3.5 shows that $e_h = O\left(h^2\right)$ all over the region $\Omega_h \cup \Gamma_h$. The support of ∂C_h is very near to Γ_h, and a refined estimation of e_h is sought in this subsection.

Lemma 4.4 *Let v_h be the solution to the problem such that $-\Delta_h v_h = 1$ in Ω_h and $v_h = 0$ on Γ_h, then*

$$\left(v_h\right)_{ij} = O\left(1\right) \cdot dist\left(\left(x_i, y_j\right), \Gamma_h\right).$$

Proof Let v be the continuous solution of $-\Delta v = 1$ in Ω and $v = 0$ on Γ. Since $-\Delta v + \Delta_h v = O\left(h\right)$ in Ω_h, for sufficiently small h, there exists a constant $1 > c > 0$, independent of h, such that

$$-\Delta_h v \geq 1 - c > 0.$$

Since $-\Delta_h \left(\frac{1}{1-c} v - v_h\right) > 0$ in Ω_h and $\frac{1}{1-c} v - v_h = 0$ on Γ_h, the maximum principle (Lemma 3.2) implies that

$$0 \le v_h \le \frac{v}{1-c}.$$

For $(x_i, y_j) \in \Omega_h$, let (x, y) be the closest point in Γ_h to (x_i, y_j), i.e. $|(x, y) - (x_i, y_j)| = dist\left((x_i, y_j), \Gamma_h\right)$, then

$$\begin{aligned} 0 \le (v_h)_{ij} \le \frac{v(x_i, y_j)}{1-c} &\le \frac{v(x,y) + O\left(|(x,y) - (x_i, y_j)|\right)}{1-c}, \quad \text{using } v(x,y) = 0 \\ &\le \frac{1}{1-c} O\left(dist\left((x_i, y_j), \Gamma_h\right)\right), \end{aligned}$$

and the lemma follows. $\square$

Lemma 4.5 *Let w_h be the solution of*

$$-\Delta_h w_h = \begin{cases} 0 & \text{in } \Omega_h^\circ \\ 1 & \text{in } \Omega_h^* \end{cases} \text{ and } w_h = 0 \text{ on } \Gamma_h.$$

Then $0 \le (w_h)_{ij} \le O(h) \cdot \min\left\{h_{i \pm \frac{1}{2}, j}, h_{i, j \pm \frac{1}{2}}\right\}$.

Proof Since $-\Delta_h w_h \ge 0$ in Ω_h, Lemma 3.2 implies that the minimum is attained on Γ_h, so $w_h \ge 0$. Similarly, since $-\Delta_h w_h = 0$ in Ω_h°, the maximum should be attained either on Ω_h^* or on Γ_h. All the values at Γ_h are the minimum, so the maximum is attained at some $(x_{i^*}, y_{j^*}) \in \Omega_h^*$. The node (x_{i^*}, y_{j^*}) has at least one neighborhood in Γ_h, let us say $(x_{i^*-1}, y_{j^*}) \in \Gamma_h$, then using the fact that $(w_h)_{i^* j^*} \ge (w_h)_{i^* \pm 1, j^*}, (w_h)_{i^*, j^* \pm 1}$ leads to

$$\begin{aligned} &\left(\frac{(w_h)_{i^* j^*} - (w_h)_{i^*+1, j^*}}{h_{i^*+\frac{1}{2}, j^*}} + \frac{(w_h)_{i^* j^*}}{h_{i^*-\frac{1}{2}, j^*}}\right) \frac{2}{h_{i^*-\frac{1}{2}, j^*} + h_{i^*+\frac{1}{2}, j^*}} \\ + &\left(\frac{(w_h)_{i^* j^*} - (w_h)_{i^*, j^*+1}}{h_{i^*, j^*+\frac{1}{2}}} + \frac{(w_h)_{i^* j^*} - (w_h)_{i^*, j^*-1}}{h_{i^*, j^*-\frac{1}{2}}}\right) \frac{2}{h_{i^*, j^*-\frac{1}{2}} + h_{i^*, j^*+\frac{1}{2}}} = 1, \end{aligned}$$

and

$$\begin{aligned} \left(\frac{(w_h)_{i^* j^*}}{h_{i^*-\frac{1}{2}, j^*}}\right) \frac{2}{h_{i^*+\frac{1}{2}, j^*} + h_{i^*-\frac{1}{2}, j^*}} &\le 1 \\ 0 \le (w_h)_{i^* j^*} \le \frac{h_{i^*+\frac{1}{2}, j^*} + h_{i^*-\frac{1}{2}, j^*}}{2} h_{i^*-\frac{1}{2}, j^*} &\le h^2. \end{aligned}$$

Therefore for all $(x_i, y_j) \in \Omega_h \cup \Gamma_h$, $0 \le (w_h)_{ij} \le h^2$, which proves the lemma for $(x_i, y_j) \in \Omega_h^\circ$ because $\min\left\{h_{i \pm \frac{1}{2}, j}, h_{i, j \pm \frac{1}{2}}\right\} = h$ in this case. Now consider the case when $(x_i, y_j) \in \Omega_h^*$. For $k = 1, \ldots, 4$, let P_k be the neighboring point of $P = (x_i, y_j)$ and $w_k = w_h(P_k)$, $h_k = |P - P_k|$. Then we have $-\Delta_h w_h = 1$ at $(x_i, y_j) \in \Omega_h^*$, which implies

$$\begin{aligned} \frac{2(h_1 h_3 + h_2 h_4)}{h_1 h_2 h_3 h_4} (w_h)_{i,j} = &\frac{2}{h_1(h_1 + h_3)} w_1 + \frac{2}{h_3(h_1 + h_3)} w_3 \\ &+ \frac{2}{h_2(h_2 + h_4)} w_2 + \frac{2}{h_4(h_2 + h_4)} w_4 + 1. \end{aligned} \tag{11}$$

Note that since $P \in \Omega_h^*$, it has at least one neighboring node in Γ_h. Also note that $h_k < h$ implies $w_k = 0$. Let $h_{min} = \min\{h_1, h_2, h_3, h_4\}$. Using the fact that $w_k \le h^2$, for $k = 1, \ldots, 4$, it is not difficult to show

$$\frac{h_1 h_2 h_3 h_4}{h_1 h_3 + h_2 h_4} \le h_{mim} h \quad \text{and} \quad \frac{1}{h_k \left(h_k + h_{k\pm 2}\right)} w_k \le \frac{h^2}{h\left(h + h_{k\pm 2}\right)} \le 1, \quad k = 1, \ldots, 4.$$

Applying these inequalities to (11), we have a bound for $(w_h)_{i,j}$

$$(w_h)_{i,j} \le 5 h h_{min},$$

which completes the proof of the lemma. $\square$

Theorem 4.6 *For each $(x_i, y_j) \in \Omega_h$,*

$$(e_h)_{ij} = O\left(h^2\right)\left(dist\left((x_i, y_j), \Gamma_h\right) + \min\left(h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right)\right).$$

Proof From Lemma 3.1, there exist constants $c_1, c_2 > 0$ such that

$$\begin{array}{lcl} -c_1 h^2 \le -\Delta_h e_h \le c_1 h^2 & \text{in} & \Omega_h^\circ \\ -c_2 h \le -\Delta_h e_h \le c_2 h & \text{in} & \Omega_h^* \end{array}.$$

Using the notations in Lemmas 4.4 and 4.5,

$$-\Delta_h\left(\left(c_1 h^2\right) v_h + \left(c_2 h - c_1 h^2\right) w_h\right) = \begin{cases} c_1 h^2 & \text{in } \Omega_h^\circ \\ c_2 h & \text{in } \Omega_h^* \end{cases}.$$

Using Lemmas 3.2, 4.4 and 4.5, we have

$$\begin{aligned} \left|(e_h)_{ij}\right| &\le \left(c_1 h^2\right)(v_h)_{ij} + \left(c_2 h - c_1 h^2\right)(w_h)_{ij} \\ &\le \left(c_1 h^2\right) O(1) \cdot dist\left((x_i, y_j), \Gamma_h\right) + \left(c_2 h - c_1 h^2\right) O(h) \cdot \min\left\{h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right\} \\ &= O\left(h^2\right)\left(dist\left((x_i, y_j), \Gamma_h\right) + \min\left(h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right)\right). \end{aligned}$$

This shows the theorem. $\square$

Corollary 4.7 *If $(x_i, y_j) \in \Omega_h^*$, $(e_h)_{ij} = O\left(h^2\right)\min\left(h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right)$.*

Proof Simply because $dist\left((x_i, y_j), \Gamma\right) \le \min\left(h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right)$.

Corollary 4.8 *If $(x_i, y_j) \in \Omega_h^\circ$ and one of its neighborhoods belong to Ω_h^*, $(e_h)_{ij} = O\left(h^3\right)$.*

Proof Simply because $\min\left(h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right) = h$ and $dist\left((x_i, y_j), \Gamma\right) \le 2h$.

Theorem 4.9 *(Supra-convergence on gradient) Let u be a continuous solution to the problem (2) and u_h a discrete solution to the problem (3). Then the ℓ_2- accuracy of the gradient of $\nabla_h u - \nabla_h u_h$ is $O(h^2)$, that is,*

$$\|\nabla_h u - \nabla_h u_h\|_{\ell_2} = \sqrt{\int\limits_{\Omega_h} \nabla_h e \cdot \nabla_h e \, d\Omega_h} = O(h^2).$$

Proof Let $e = u - u_h$ and substitute u and v with e for (10), then we obtain

$$\int\limits_{\Omega_h} \nabla_h e \cdot \nabla_h e \, d\Omega_h = -\int\limits_{\Omega_h} (\Delta_h e)\, e \, d\Omega_h + \int\limits_{\partial C_h} (n \cdot \nabla_h e)\, e \, d\Gamma_h. \tag{12}$$

Lemma 4.3 shows that the first integral amounts to $O\left(h^4\right)$, and it is enough to consider the second integral. In the definition of the line integral (e.g., (9)).

$$\begin{aligned}
&\int\limits_{\partial C_h} \left(D_h^x e_h\right) e_h \left(n \cdot e_1\right) d\Gamma \\
&= \sum_{(x_i,y_j)\in\Omega_h} \begin{pmatrix} + & \frac{(e_h)_{i+1,j}-(e_h)_{i,j}}{h_{i+\frac{1}{2},j}} \frac{(e_h)_{i+1,j}+(e_h)_{i,j}}{2} \\ - & \frac{(e_h)_{ij}-(e_h)_{i-1,j}}{h_{i-\frac{1}{2},j}} \frac{(e_h)_{i,j}+(e_h)_{i-1,j}}{2} \end{pmatrix} \frac{h_{i,j+\frac{1}{2}} + h_{ij-\frac{1}{2}}}{2} \\
&= \sum_{\substack{(x_i,y_j)\in\Omega_h \\ \text{or } (x_{i+1},y_j)\in\Omega_h}} \frac{(e_h)_{i+1,j}-(e_h)_{ij}}{h_{i+\frac{1}{2},j}} \frac{(e_h)_{i+1,j}+(e_h)_{ij}}{2} \begin{pmatrix} + & \frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} \\ - & \frac{h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{2} \end{pmatrix},
\end{aligned}$$

all the non-zero terms in the summation appear only where $(x_i, y_j) \in \Omega_h^*$ or $(x_{i+1}, y_j) \in \Omega_h^*$; in the other cases, $\left(h_{i,j+\frac{1}{2}} + h_{i,j-\frac{1}{2}}\right) - \left(h_{i+1,j+\frac{1}{2}} + h_{i+1,j-\frac{1}{2}}\right) = (2h) - (2h) = 0$. When$(x_i, y_j) \in \Omega_h^*$ or $(x_{i+1}, y_j) \in \Omega_h^*$, $dist\left((x_i, y_j), \Gamma_h\right) \le 2h$ and $dist\left((x_{i+1}, y_j), \Gamma_h\right) \le 2h$, and Corollaries 4.7 and 4.8 state that $\frac{e_{ij}+e_{i+1,j}}{2} = O(h^3)$ and $\frac{e_{i+1,j}-e_{ij}}{h_{i+\frac{1}{2},j}} = O(h^2)$. Combining the estimates,

$$\begin{aligned}
\int_{\partial c_h} (D_h^x e) e (n \cdot e_1) d\Gamma = O(h^5) &\sum_{\substack{(x_i,y_j)\in\Omega_h^* \\ \text{or } (x_{i+1},y_j)\in\Omega_h^*}} \frac{(h_{i,j+\frac{1}{2}} + h_{i,j-\frac{1}{2}}) - (h_{i+1,j+\frac{1}{2}} + h_{i+1,j-\frac{1}{2}})}{2} \\
&= O(h^5) \cdot O(h^{-1}) \cdot O(h) = O(h^5)
\end{aligned}$$

Here, we use the fact that the number of elements in Ω_h^* is $O(h^{-1})$, since Γ is one dimensional. Repeating the same process on the other term in $\int_{\partial c_h} (n \cdot \nabla_h e) e \, d\Gamma = \int_{\partial c_h} (D_h^x e) e (n \cdot e_1) d\Gamma + \int_{\partial c_h} (D_h^y e) e (n \cdot e_2)\, d\Gamma$ completes the proof. $\square$

5 Numerical Test

The linear system was solved by the ILU-preconditioned BiCGSTAB method [17] with stopping criteria on residual $\|r^n\| < 10^{-10} \left\|r^0\right\|$. The error and its gradient in the L^2 norm are calculated by the formula in the section of discrete divergence theorem.

Example 5.1 *(Poisson equation in two dimensions)*

Assume $\Omega \subset \mathbb{R}^2$ to be a circle of center $(0, 0)$ and radius 1. Choose $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ such that $u\left(x, y\right) = \frac{y}{(x+2)^2+y^2}$ is the exact solution of the problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \Gamma. \end{aligned}$$

grid	$\|u-u_h\|_{L^\infty}$	order	$\|u-u_h\|_{L^2}$	order	$\|\nabla u-\nabla_h u_h\|_{L^2}$	order
40^2	1.28×10^{-4}		9.52×10^{-5}		6.08×10^{-4}	
80^2	3.35×10^{-5}	1.93	2.45×10^{-5}	1.96	1.66×10^{-4}	1.87
160^2	8.54×10^{-6}	1.97	6.24×10^{-6}	1.97	4.35×10^{-5}	1.93
320^2	2.16×10^{-6}	1.98	1.57×10^{-6}	1.99	1.11×10^{-5}	1.97

Table 1: Convergence rate for the Poisson problem in two dimensions, example 5.1

grid	$\|u-u_h\|_{L^\infty}$	order	$\|u-u_h\|_{L^2}$	order	$\|\nabla u-\nabla_h u_h\|_{L^2}$	order
20^3	2.22×10^{-3}		1.34×10^{-3}		6.08×10^{-3}	
40^3	5.63×10^{-4}	1.97	3.38×10^{-4}	1.98	1.72×10^{-3}	1.98
80^3	1.40×10^{-4}	2.00	8.41×10^{-5}	2.00	4.29×10^{-4}	2.00
160^3	3.48×10^{-5}	2.00	2.09×10^{-5}	2.00	1.07×10^{-4}	2.00

Table 2: Convergence rate for the Poisson problem in three dimensions, example 5.2

Table 1 confirms our theoretical results that the numerical solution and its gradient are both second order accurate.

Example 5.2 *(Poisson equation in three dimensions)*

Assume $\Omega\subset\mathbb{R}^3$ to be a sphere of center $(0,0,0)$ and radius 1. With exact solution $u(x,y,z)=\frac{\exp^{-(x^2+y^2+z^2)}}{(2+x)^2+y^2}$, choose $f{:}\Omega\to\mathbb{R}$ and $g:\Gamma\to\mathbb{R}$ accordingly as the previous example. Table 2 shows that the numerical solution and its gradient are both second order accurate.

Example 5.3 *(Helmholtz-Hodge projection)*

In this example, we consider an important application of the Shortley-Weller method on fluid flow with free surface. The incompressible Navier-Stokes equations consist of momentum equation and incompressibility-constraint equation, and can be written as the momentum equation without pressure term applied with the Hodge-Helmotz projection. A vector field U^* is uniquely decomposed into a sum of divergence-free vector field U and gradient field ∇p. The Hodge-Helmotz projection of U^* takes the divergence-free vector field dropping the gradient field in the decomposition. In this example, we implement the projection in the discrete setting by applying the Shortley-Weller method on the following Poisson equation.

$$\begin{aligned} -\Delta_h p_h &= \nabla_h\cdot U^* && \text{in } \Omega_h \\ p_h &= 0 && \text{on } \Gamma_h \end{aligned}$$

At the free surface, Dirichlet boundary condition is imposed [15]. The projection of U^* is calculated as $U_h=U^*-\nabla_h P_h$. For the test, we take $\Omega=\left\{(x,y)\in\left[-\frac{3}{2},\frac{3}{2}\right]\times\left[-\frac{3}{2},\frac{3}{2}\right]:\left(\frac{x}{1.1}\right)^2+\left(\frac{y}{0.8}\right)^2\le1\right\}$ and $U^*=\left(\cos(\pi x)\sin(\pi y),-\sin(\pi x)\cos(\pi y)\right)+\nabla\left(e^{y-x^2}\left(\left(\frac{x}{1.1}\right)^2+\left(\frac{y}{0.8}\right)^2-1\right)\right)$. Table 3 shows that the approximation U_h is second order accurate, which is due to the second order convergence of $\nabla_h p_h$.

6 Conclusion

We have introduced the proof that the solution gradient of the Shortley-Weller method is second order accurate in general domains. For the proof, we presented the new estimates for e_h and the novel discrete divergence theorem suited in the discrete setting of the Shortley-Weller method.

grid	$\|U - U_h\|_{L^2}$	order
40^2	6.85×10^{-3}	
80^2	1.73×10^{-4}	1.98
160^2	4.34×10^{-5}	1.99
320^2	1.08×10^{-5}	2.00

Table 3: Convergence rate for the Hodge-Helmotz projection, example 5.3

Our proof was presented only in two dimensions, but its extension to three dimensions would be a line-by-line substitution, which we omit and put off to a future work. A thorough numerical test in [13] suggest that the solution gradient is second order accurate not only in L^2 but also in L^∞. Our current article proved the former only and we plan to discuss the latter issue in future work.

References

[1] J.H. Bramble and B.E. Hubbard. On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation. *Numer. Math.*, 4:313–3217, 1962.

[2] X. Chen, N. Matsunaga, and T. Yamamoto. Smoothing newton methods for nonsmooth Dirichlet problems. *Kluwer Academic Publishers*, pages 65–79, 1998.

[3] P. Ciarlet. *Introduction to numerical linear and optimization.* Cambridge Texts in Applied Mathematics, 40 West 20th street, New York, NY 10011, 1998.

[4] Q Fang and T Yamamoto. Superconvergence of finite difference approximations for convection-diffusion problems. *Numer. Linear Algebra Appl.*, 2(8):99–110, 2001.

[5] F. Gibou, R. Fedkiw, L.-T. Cheng, and M. Kang. A second–order–accurate symmetric discretization of the Poisson equation on irregular domains. *J. Comput. Phys.*, 176:205–227, 2002.

[6] F. Harlow and J. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluids with free surfaces. *Phys. Fluids*, 8:2182–2189, 1965.

[7] B. Heimsund, X. Tai, and J. Wang. Superconvergence for the gradient of finite element approximations by l^2 projections. *SIAM J. Numer. Anal.*, 40:1263–1280, 2002.

[8] M. Krizek and P. Neittaanmaki. On a global superconvergence of the gradient of linear triangular elements. *J. Comput. Appl. Math.*, 18:221–233, 1987.

[9] M. Krizek and P. Neittaanmaki. On superconvergence techniques. *Acta Appl. Math.*, 9:175–198, 1987.

[10] Z.C. Li, H.Y. Hu, S. Wang, and Q. Fang. Superconvergence of solution derivatives of the Shortley-Weller difference approximation to Poisson's equation with singularities on polygonal domains. *Appl. Numer. Math.*, 58:689–704, 2008.

[11] Z.C. Li, H.Y. Hu, Q. Fang, and T. Yamamoto. Superconvergence of solution derivatives for the Shortley-Weller difference approximation of Poisson's equation. ii. singularity problems. *Numer. Func. Anal. Optimiz.*, 24:195–221, 2003.

[12] N. Matsunaga and T. Yamamoto. The fast solution of Poisson's and the biharmonic equations on irregular regions. *J. Comput. Appl. Math.*, 116:263–273, 2000.

[13] Y.-T. Ng, H. Chen, C. Min, and F. Gibou. Guidelines for Poisson solvers on irregular domains with Dirichlet boundary conditions using the ghost fluid method. *J. Sci. Comput.*, 41(2):300–320, May 2009.

[14] Y.-T. Ng, C. Min, and F. Gibou. An efficient fluid–solid coupling algorithm for single-phase flows. *J. Comput. Phys.*, 228:8807–8829, 2009.

[15] C. Pozrikidis. Introduction to theoretical and computational fluid dynamics. *Oxford university press*, 1997.

[16] J.W. Purvis and J.E. Burkhalter. Prediction of critical mach number for store configurations. *AIAA J.*, 17:1170–1177, 1979.

[17] Y. Saad. *Iterative methods for sparse linear systems.* PWS Publishing, 1996.

[18] G.H. Shortley and R. Weller. Numerical solution of laplace's equation. *J. Appl. Phys.*, 9:334–348, 1938.

[19] J.C. Strikwerda. Finite difference schemes and partial differential equations. *SIAM*, 2004.

[20] G. Yoon and C. Min. A review of the supra-convergences of Shortley-Weller method for Poisson Equation. *Kor. Soc. Ind. Appl. Math.*, 18:51–60, 2014.