Analyses on the Finite Difference Method by Gibou *et al.* for Poisson equation

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Abstract

Gibou *et al.* in [4] introduced a finite difference method for solving the Poisson equation in irregular domains with the Dirichlet boundary condition. Contrary to its great importance, its properties have not been mathematically analyzed, but have just been numerically observed. In this article, we present two analyses for the method. One proves that its solution is second order accurate, and the other estimates the condition number of its linear system. According to our estimation, the condition number of the unpreconditioned linear system is of size $O\left(1/\left(h \cdot h_{min}\right)\right)$, and each of Jacobi, SGS, and ILU preconditioned systems is of size $O(h^{-2})$. Furthermore, our analysis shows that the condition number of MILU is of size $O(h^{-1})$, the most successful one.

1 Introduction

The Shortley-Weller method [13] is a basic finite difference method for solving the Poisson equation with the Dirichlet boundary condition. It is a simple sum of the central finite differences in the Cartesian directions. Though implemented in uniform grid, the method can handle arbitrarily shaped domains. Its solution is second order accurate to the analytic solution. Usually the gradient of a second order accurate solution is only first order accurate, however the solution exhibits a supra-convergence behaviour. Its gradient is also second order accurate [10].

Though its excellence in efficiency and accuracy, the Shortley-Weller method constitutes a non-symmetric linear system. Only in one dimension, the linear system can be cast in a symmetric form [14]. Since the Laplacian is self-adjoint, the method that approximates the operator is expected to be symmetric. Gibou *et al.* [4] introduced a simple modification of the Shortley-Weller method that results in a symmetric linear system. Numerical tests [10] suggest that the solution is still second order accurate.

Compared to the Shortley-Weller method, the method by Gibou *et al.* has an advantage to solve symmetric linear system. The gain, however, turns out to have not come free. The supra-convergence of the Shortley-Weller method is lost with the gain. The solution gradient is only first order accurate. Both methods have their own pros and cons as described above, and a choice between them depends on the characteristic of the given problem. For example, in application to incompressible fluid flows the solution gradient is a physical variable and the Shortley-Weller method would be preferred, and in application to heat flows the method by Gibou *et al.* would be desired.

Contrary to their great importance, the convergence properties of the Shortley-Weller method and the method by Gibou *et al.* have just been numerically observed. The numerical tests, which are merely finite however many, are not enough to ascertain the properties. Though the Shortley-Weller method [13] was introduced in 1938, it is very recent to see some mathematical analyses on the gradient

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of its solution. In 2003, Li *et al.* [7, 9] showed the second order accuracy in rectangular domains, and Li *et al.* [8] the 1.5 order accuracy in polygonal domains. In 2014, we in [16] showed the second order accuracy in general domains.

An important aspect of a Poisson solver is the size of the condition number of its associated linear system. A large-sized condition number not only delays the convergence to solve but also drops many significant digits in the approximation. The seminal work of Gustafsson [6] shows that only the modified incomplete-LU (MILU) preconditioner among many incomplete-LU (ILU) type preconditioners enhances the condition number of the standard finite difference Poisson solver with different order of magnitude. His work can deal only with rectangular domains. It is also very recent to see such estimations in irregular domains. We in [15] arrived at the same conclusion for the Shortley-Weller method. Only the MILU enhances the condition number with different order of growth with respect to grid step size h.

In this article, we introduce two analyses for the method by Gibou *et al.* One proves that the solution is second order accurate to the analytic solution. The other estimates the condition number of its linear system with and without preconditioners. The estimation shows that the unpreconditioned linear system has a very large condition number of size $O\left(1/\left(h \cdot h_{min}\right)\right)$, where h is the default step size of uniform grid and h_{min} is the minimum step size that is usually much smaller than h. We then show that Jacobi, symmetric Gauss-Seidel (SGS), and ILU preconditioners on the linear system reduce the condition number from $O\left(1/\left(h \cdot h_{min}\right)\right)$ to $O\left(h^{-2}\right)$. Finally, we show that MILU preconditioner exels the others by gaining $O\left(h^{-1}\right)$ size.

2 Convergence analysis

Consider a uniform grid $h\mathbb{Z}^2$ with step size h. Let Ω_h be the set of nodes of the grid belonging to Ω, and Γ_h be the set of intersection points between Γ and grid lines. A grid node $(x_i, y_i) \in \Omega_h$ has four neighboring nodes in $\Omega_h \cup \Gamma_h$, $(x_{i\pm1}, y_j)$ and $(x_i, y_{j\pm1})$ in $\Omega_h \cup \Gamma_h$, as illustrated in Figure 1. Let $h_{i+\frac{1}{2},j}$ denote the distance from (x_i, y_j) to its neighbor (x_{i+1}, y_j). Other distances $h_{i-\frac{1}{2},j}, h_{i,j+\frac{1}{2}}, h_{i,j-\frac{1}{2}}$ are similarly defined. The work of Gibou *et al.* [4] solves the Poisson equation with Dirichlet boundary condition

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in} \quad \Omega \\ u & = & g & \text{on} \quad \Gamma, \end{array} \right. \tag{1}$$

by solving the discrete equation

$$\left\{ \begin{array}{rcll} -\Delta_h u_h\left(x_i, y_j\right) & = & f\left(x_i, y_j\right), & \left(x_i, y_j\right) \in \Omega_h \\ u_h\left(x_i, y_j\right) & = & g\left(x_i, y_j\right), & \left(x_i, y_j\right) \in \Gamma_h. \end{array} \right. \tag{2}$$

Here the discrete Laplacian operator $\Delta_h u : \Omega_h \to \mathbb{R}$ is defined as

$$-\left(\Delta_h u\right)_{ij} := \left(\frac{u_{ij} - u_{i+1,j}}{h_{i+\frac{1}{2},j}} + \frac{u_{ij} - u_{i-1,j}}{h_{i-\frac{1}{2},j}}\right)\frac{1}{h} + \left(\frac{u_{ij} - u_{i,j+1}}{h_{i,j+\frac{1}{2}}} + \frac{u_{ij} - u_{i,j-1}}{h_{i,j-\frac{1}{2}}}\right)\frac{1}{h}. \tag{3}$$

The equations for each node point (x_i, y_j) constitute a symmetric linear system whose matrix is an M-matrix. It was numerically observed in [10] that the numerical solution is second order accurate and the gradient of the solution is only first order accurate.

In this section, we analyze the consistency and convergence accuracy of the Gibou *et al.* method, which shows that the discrete solution approximates the continuous solution with the second order accuracy. Though the consistency order of the discretization ranges from the zero to the second, its convergence order is the second order everywhere.

Definition 2.1 *$\Omega_h^* \subset \Omega_h$ denotes the set of grid nodes adjacent to Γ_h, and $\Omega_h^\circ = \Omega_h \setminus \Omega_h^*$.*

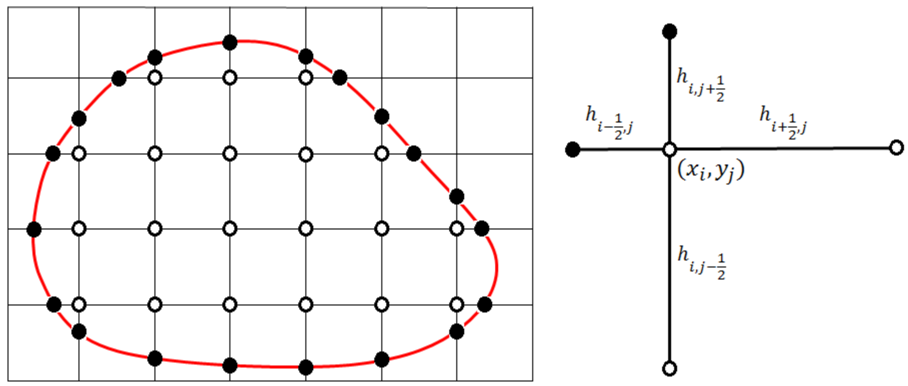


Figure 1: Grid nodes in Ω_h are marked by $\circ$ and nodes in Γ_h by $\bullet$. A grid node $(x_i, y_j) \in \Omega_h$ has four neighboring nodes in $\Omega_h \cup \Gamma_h$.

In Definition 2.1, we divides the nodes in Ω_h into two sets. Every node $(x_i, y_j) \in \Omega_h^\circ$ has the four neighboring points inside Ω_h so that $h_{i\pm\frac{1}{2},j} = h_{i,j\pm\frac{1}{2}} = h$. On the other hand, if $(x_i, y_j) \in \Omega_h^*$, then at least one of its four neighboring points belongs to Γ_h.

Lemma 2.2 (Consistency error) *For a smooth function $u : \Omega \to \mathbb{R}$,*

$$|\Delta_h u - \Delta u| \leq \begin{cases} C_1 h^2, & \text{in } \Omega_h^\circ \\ C_2 + C_3 h, & \text{in } \Omega_h^*, \end{cases} \tag{4}$$

where $C_1, C_2,$ and C_3 are constants independent of h.

Proof A simple Taylor series expansion on u shows the consistency error (4) with constants C_1, C_2, and C_3 dependent only on u and Ω. $\square$

The discrete equation (2) for each $(x_i, y_j) \in \Omega_h$ forms a symmetric linear system whose matrix is an M-matrix [12]. An important property of an M-matrix is that its inverse is non-negative in every entry, from which the discrete maximum principle follows.

Lemma 2.3 (Discrete maximum principle) *If $-\Delta_h \nu \geq 0$, then the minimum value of ν should be achieved on Γ_h. Similarly, if $-\Delta_h \nu \leq 0$, then the maximum value of ν should be achieved on Γ_h. Likewise, if $-\Delta_h \nu_1 \geq -\Delta_h \nu_2$ in Ω_h and $\nu_1 \geq \nu_2$ on Γ_h, then $\nu_1 \geq \nu_2$ on $\Omega_h \cup \Gamma_h$.*

Lemma 2.4 *Let w_h be the solution of*

$$-\Delta_h w_h = \begin{cases} 0 & \text{in } \Omega_h^\circ \\ 1 & \text{in } \Omega_h^* \end{cases} \quad \text{and } w_h = 0 \text{ on } \Gamma_h.$$

Then $0 \leq w_h \leq h^2$ in Ω_h.

Proof Since $-\Delta_h w_h \geq 0$ in Ω_h and $w_h = 0$ on Γ_h, the maximum principle implies that $w_h \geq 0$ in Ω_h. Furthermore, the maximum of w_h is attained at some point $(x_{i^*}, y_{j^*}) \in \Omega_h^*$. Belonging to Ω_h^*, at least one of the four neighborhood points of (x_{i^*}, y_{j^*}), say (x_{i^*-1}, y_{j^*}), is a boundary point. Since all the terms in $-\Delta_h w_h(x_{i*}, y_{j^*})$ are nonnegative and $w_h(x_{i^*-1}, y_{j^*}) = 0$, we have

$$\frac{w_h(x_{i^*}, y_{j^*})}{h h_{i^*-\frac{1}{2},j^*}} \leq -\Delta_h w_h(x_{i^*}, y_{j^*}) = 1, \quad \text{or} \quad w_h(x_{i^*}, y_{j^*}) \leq h h_{i^*-\frac{1}{2},j^*} \leq h^2,$$

which proves the lemma. $\square$

Lemma 2.5 *Let v_h be the solution of*

$$-\Delta_h v_h = \begin{cases} 1 & \text{in } \Omega_h^\circ \\ 0 & \text{in } \Omega_h^* \end{cases} \quad \text{and } v_h = 0 \text{ on } \Gamma_h.$$

Then $0 \le v_h \le C_v$ in Ω_h for sufficiently small h, where C_v is independent of h.

Proof Since $-\Delta_h v_h \ge 0$ in Ω_h and $v_h = 0$ on Γ_h, the maximum principle implies that $v_h \ge 0$ in Ω_h. Consider an analytic solution $v : \Omega \to \mathbb{R}$ satisfying $-\Delta v = 2$ in Ω and $v = 0$ on Γ. Lemma 2.2 implies that for sufficiently small h, we have

$$-\Delta_h\big(v - v_h\big) = \begin{cases} -\Delta_h v - 1 & \ge & 0, & \text{in } \Omega_h^\circ \\ -\Delta_h v & \ge & -\widetilde{C}, & \text{in } \Omega_h^*, \end{cases}$$

with some constant $\widetilde{C} > 0$. Using the discrete function w_h given in Lemma 2.4, we have an inequality

$$-\Delta_h\big(v - v_h + \widetilde{C} w_h\big) \ge 0 \quad \text{in } \Omega_h.$$

Since $v - v_h + \widetilde{C} w_h = 0$ on Γ_h, we have $v - v_h + \widetilde{C} w_h \ge 0$. This inequality and Lemma 2.4 imply that

$$0 \le v_h \le v + \widetilde{C} h^2.$$

Taking $C_v = \max|v| + \widetilde{C}$ gives the estimate $0 \le v_h \le C_v$ for some constant C independent of h. □

Theorem 2.6 *Let u be a continuous solution to the problem* (1) *and u_h a discrete solution to the problem* (2)*. Then we have*

$$|u - u_h| = O(h^2) \quad \text{in } \Omega_h.$$

Proof Using Lemmas 2.4 and 2.5, the consistency lemma reads

$$|\Delta_h u - \Delta u| \le C_1 h^2(-\Delta_h v_h) + (C_2 + C_3 h)(-\Delta_h w_h).$$

On the other hand, since $\Delta u = \Delta_h u_h = f$ in Ω_h, we have

$$\begin{array}{rcl} -\Delta_h\big(C_1 h^2 v_h + (C_2 + C_3 h) w_h - (u - u_h)\big) & \ge & 0 \\ -\Delta_h\big(C_1 h^2 v_h + (C_2 + C_3 h) w_h + (u - u_h)\big) & \ge & 0. \end{array}$$

Since $v_h = w_h = u - u_h = 0$ on Γ_h, the maximum principle implies that

$$\begin{array}{rcl} |u - u_h| & \le & C_1 h^2 v_h + (C_2 + C_3 h) w_h \\ & \le & C_1 C_v h^2 + (C_2 + C_3 h)) h^2 = h^2(C_1 C_v + C_2 + C_3 h), \end{array}$$

which shows the convergence estimate $|u - u_h| = O(h^2)$ in Ω_h. □

3 Condition number of the preconditioned matrices

In this section, we consider the application of basic preconditioning techniques to the linear system that is associated with the Gibou *et al.* method. Before we proceed to the discussion of the preconditioners, we provide the estimation of the condition number of the linear system associated with the Gibou *et al.* method.

We may assume that the domain Ω is a subset of $\{(x,y)\mathbb{R}^2 : 0 \le x \le a,\ 0 \le y \le b\}$. Let $\Omega_h := \{(ih, jh) \in \Omega : 1 \le i \le N, 1 \le j \le M\}$ with the lexicographical order on Ω_h [3]. Let $K := |\Omega_h|$ and $\mathbf{x}_k := (i_k h, j_k h) \in \Omega_h$ for $k = 1, \ldots, K$ according to the order. Throughout this section, let A be the $K \times K$ matrix corresponding to the discrete Poisson equation (2), which is symmetric and positive definite. The entry $a_{r,s}$ of A with $\mathbf{x}_r = (i_r h, j_r h)$ and $\mathbf{x}_s = (i_s h, j_s h)$ are given as

$$a_{r,s} = \begin{cases} -\frac{1}{h^2} & \text{if } i_s = i_r \pm 1 \text{ and } j_r = j_s \\ \frac{1}{h}\left(\frac{1}{h_{i_r+\frac{1}{2},j_r}} + \frac{1}{h_{i_r-\frac{1}{2},j_r}} + \frac{1}{h_{i_r,j_r+\frac{1}{2}}} + \frac{1}{h_{i_r,j_r-\frac{1}{2}}}\right) & \text{if } s = r \\ -\frac{1}{h^2} & \text{if } i_s = i_r \text{ and } j_s = j_r \pm 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where $h_{i_r\pm\frac{1}{2},j_r}$ and $h_{i_r,j_r\pm\frac{1}{2}}$ are given as

$$h_{i_r+\frac{1}{2},j_r} = \begin{cases} h, & \text{if } i_{r+1} = i_r + 1 \\ |i_r h - x_\Gamma|, & \text{if } \exists \mathbf{x}_\Gamma = (x_\Gamma, y_\Gamma) \in \Gamma \text{ such that } i_r h < x_\Gamma < (i_r+1)h \end{cases} \tag{6}$$

and the others are given in the same fashion.

Theorem 3.1 *Let λ be an eigenvalue of A, then $0 < C \le \lambda \le \frac{8}{h \cdot h_{min}}$ for some $C > 0$ independent of h and $h_{min} = \min_{(ih,jh)\in\Omega_h} \left\{h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right\}$.*

Proof Let λ be an eigenvalue of A. Then λ is a positive real number because A is symmetric positive definite. In order to find an upper bound of λ, we apply the Gerschgorin Circle Theorem. Since A is diagonally dominant, the Gerschgorin Circle Theorem implies that

$$\lambda \le a_{k,k} + \sum_{j\neq k} |a_{k,j}| \le 2a_{k,k}$$

for some $k = 1, \ldots, K$. From (5), we obtain that $a_{k,k} \le \frac{4}{h \cdot h_{min}}$ for all $k = 1, \ldots, K$, which gives an upper bound $\frac{8}{h \cdot h_{min}}$ for λ. On the other hand, applying the Perron-Frobenius Theorem to the M-matrix A^{-1} shows that the smallest eigenvalue λ_{min} of A is simple and has a positive eigenvector $u \in \mathbb{R}^K$. We may assume $\max_{i=1,\ldots,K} u_i = 1$. Regarding u as a discrete function defined on $\Omega_h \cup \Gamma_h$ with $u = 0$ on Γ_h, we can see that with the help of w_h and v_n given in Lemmas 2.4 and 2.5, we have

$$-\Delta_h u = \lambda_{min} u \le \lambda_{min}\big[-\Delta_h(v_h + w_h)\big] \quad \text{in } \Omega_h.$$

And the maximum principle implies the inequality $u \le \lambda_{min}(v_h + w_h)$ in Ω_h. Applying Lemmas 2.4 and 2.5, we finally obtain

$$1 = \max_{\Omega_h} u \le \lambda_{min} \max_{\Omega_h}(v_h + w_h) \le \lambda_{min}(C_v + h^2).$$

Since we may assume $h < 1$, we conclude that $\lambda_{min} \ge C > 0$ for a constant C independent of h, which completes the proof. $\square$

We have shown that the condition number of the matrix A stemmed from the linear system (2) is bounded by $O(1/(h \cdot h_{min}))$. Table 1 suggests that the bound is tight. So the smaller h_{min} becomes, the worse the condition number of A grows. Indeed, the following theorem proves that the lower and upper bounds are tight.

Theorem 3.2 *Let λ_{min} and λ_{max} be the smallest and largest eigenvalues of A in magnitude, respectively. Then $\lambda_{min} < 2\pi^2$ and $\lambda_{max} > \frac{1}{h \cdot h_{min}}$. Therefore we have $\kappa(A) = O\left(\frac{1}{h \cdot h_{min}}\right)$.*

Proof At first, we shall show that $\lambda_{max} > \frac{1}{h \cdot h_{max}}$. Let $P = (x_i, y_j)$ be the grid node nearest to the boundary so that $h_{min} = \min\left\{h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right\}$. Then take a vector $e_P \in \mathbb{R}^{|\Omega_h|}$ of which the element corresponding to P is one and the other elements are all zero. Taking a Raleigh quotient, we have

$$\lambda_{max} = \max_{0\neq v\in\mathbb{R}^{|\Omega_h|}} \frac{\langle Av, v\rangle}{\langle v, v\rangle} \geq \frac{\langle Ae_P, e_P\rangle}{\langle e_P, e_P\rangle} = \frac{1}{h}\left(\frac{1}{h_{i-\frac{1}{2},j}} + \frac{1}{h_{i+\frac{1}{2},j}} + \frac{1}{h_{i,j-\frac{1}{2}}} + \frac{1}{h_{i,j+\frac{1}{2}}}\right) > \frac{1}{h \cdot h_{min}}.$$

In order to show the estimate for λ_{min}, we choose a rectangle $R \subset \Omega$ whose boundary is aligned with the grid lines. Let $R_h = R \cap \Omega_h$. Any vector $v \in \mathbb{R}^{|R_h|}$ can be extended to $\tilde{v} \in \mathbb{R}^{|\Omega_h|}$ by taking zero values outside R_h. Let B be the associated matrix of the five-point finite difference method. Then $A\tilde{v}|_{R_h} = Bv$ and $A\tilde{v}|_{\Omega_h\setminus R_h} = 0$, which implies $\langle A\tilde{v}, \tilde{v}\rangle = \langle Bv, v\rangle$ and

$$\lambda_{min} = \min_{0\neq u\in\mathbb{R}^{|\Omega_h|}} \frac{\langle Au, u\rangle}{\langle u, u\rangle} \leq \min_{0\neq v\in\mathbb{R}^{|R_h|}} \frac{\langle A\tilde{v}, \tilde{v}\rangle}{\langle \tilde{v}, \tilde{v}\rangle} = \min_{0\neq v\in\mathbb{R}^{|R_h|}} \frac{\langle Bv, v\rangle}{\langle v, v\rangle}.$$

In [5], $\lambda_{min}(B) = \min_{0\neq v\in\mathbb{R}^{|R_h|}} \frac{\langle Bv,v\rangle}{\langle v,v\rangle}$ is exactly given as $8\frac{\sin^2\left(\frac{\pi h}{2}\right)}{h^2}$ which is less than $2\pi^2$. Combining the results of Theorem 3.1, we have $\kappa(A) = O\left(\frac{1}{h \cdot h_{min}}\right)$, which completes the proof. □

3.1 Jacobi preconditioning

We decompose A as

$$A = L + D + U$$

where L, D, and $U = L^T$ are the diagonal, the strict lower triangular, and the upper triangular parts of A, respectively. The Jacobi preconditioner is the diagonal matrix D whose diagonal entries are the same as A. The Jacobi preconditioning on the linear system results in $D^{-1}Au = D^{-1}b$. The preconditioning is, in other words, to scale each equation so that its diagonal entry becomes one. Applying the Jacobi preconditioning to its linear equation, the Gibou *et al.* method now reads

$$\begin{aligned}
u_{ij} &- \frac{\frac{1}{h_{i+\frac{1}{2},j}}}{\frac{1}{h_{i+\frac{1}{2},j}} + \frac{1}{h_{i-\frac{1}{2},j}} + \frac{1}{h_{i,j+\frac{1}{2}}} + \frac{1}{h_{i,j-\frac{1}{2}}}} u_{i+1,j} \\
&- \frac{\frac{1}{h_{i-\frac{1}{2},j}}}{\frac{1}{h_{i+\frac{1}{2},j}} + \frac{1}{h_{i-\frac{1}{2},j}} + \frac{1}{h_{i,j+\frac{1}{2}}} + \frac{1}{h_{i,j-\frac{1}{2}}}} u_{i-1,j} \\
&- \frac{\frac{1}{h_{i,j+\frac{1}{2}}}}{\frac{1}{h_{i+\frac{1}{2},j}} + \frac{1}{h_{i-\frac{1}{2},j}} + \frac{1}{h_{i,j+\frac{1}{2}}} + \frac{1}{h_{i,j-\frac{1}{2}}}} u_{i,j+1} \\
&- \frac{\frac{1}{h_{i,j-\frac{1}{2}}}}{\frac{1}{h_{i+\frac{1}{2},j}} + \frac{1}{h_{i-\frac{1}{2},j}} + \frac{1}{h_{i,j+\frac{1}{2}}} + \frac{1}{h_{i,j-\frac{1}{2}}}} u_{i,j-1} \quad = \quad \frac{hf_{i,j}}{\frac{1}{h_{i+\frac{1}{2},j}} + \frac{1}{h_{i-\frac{1}{2},j}} + \frac{1}{h_{i,j+\frac{1}{2}}} + \frac{1}{h_{i,j-\frac{1}{2}}}}.
\end{aligned} \tag{7}$$

It can be observed from the equation (7) that the eigenvalue estimation for the Jacobi-preconditioned matrix is almost independent of $h_{min} = \min_{(ih,jh)\in\Omega_h}\left\{h_{i\pm\frac{1}{2},j}, h_{i,j\pm\frac{1}{2}}\right\}$, while that for the original matrix is dependent. Thus, the presence of grid nodes too near the boundary is not problematic in the Jacobi-preconditioned matrix. Precisely, let a grid node (x_i, y_j) be very near the boundary to the left. As it gets nearer and nearer, $h_{i-\frac{1}{2},j} \to 0$ and the discretization becomes

$$\begin{aligned}
u_{ij} - 0 \cdot u_{i+1,j} - 1 \cdot g(x_{i-1}, y_j) - 0 \cdot u_{i,j-1} - 0 \cdot u_{i,j+1} &= 0 \cdot f_{ij}, \text{ or} \\
u_{ij} &= g(x_{i-1}, y_j).
\end{aligned}$$

So the equation makes a diagonal block split from the matrix and the eigenvalue of the diagonal block is one. Hence, the presence of grid nodes very near the boundary actually makes rather a benign effect on Jacobi-preconditioned matrix, contrary to its bad effect on the original matrix. Now, we prove the observation as follows.

Theorem 3.3 *For any eigenvalue λ of the Jacobi-preconditioned matrix, we have*

$$0 < \frac{h^2}{4C_v + \frac{4h^3}{h_{min}}} \leq \lambda \leq 2. \tag{8}$$

Furthermore, if $h_{min} \geq h^3$, then we have $0 < Ch^2 \leq \lambda \leq 2$ for some constant $C = C(\Omega)$.

Proof Let λ be an eigenvalue of $D^{-1}A$ and $u \in \mathbb{C}^K$ its corresponding eigenvector. Since all the diagonal elements of D are positive, let $D^{\frac{1}{2}}$ denote the square root matrix of D. Then we can see that $D^{-1}Au = \lambda u$ if and only if $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}D^{\frac{1}{2}}u = \lambda D^{\frac{1}{2}}u$. Thus, the positive definiteness of $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ verifies that $\lambda > 0$ and $u \in \mathbb{R}^K$. Since $D^{-1}A$ is diagonally dominant and $\sum_{j=1}^{K} |a_{ii}^{-1}a_{ij}| \leq 2$, the Gerschgorin Circle Theorem implies $\lambda \leq 2$.

On the other hand, we may assume that $\max_{i=1,\ldots,K} u_i = 1$. From (5), we have

$$Au = \lambda Du \ \leq \begin{cases} \lambda\frac{4}{h^2}, & \text{in } \Omega_h^\circ \\ \lambda\frac{4}{hh_{min}}, & \text{in } \Omega_h^* \end{cases} \tag{9}$$

Regarding u as a discrete function on Ω_h with $u = 0$ on Γ_h, we have $-\Delta_h u = Au$ in Ω_h, and using w_h and v_h in Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned} -\Delta_h u &\leq \lambda\tfrac{4}{h^2}\big(-\Delta_h v_h\big) + \lambda\tfrac{4}{hh_{min}}\big(-\Delta_h w_h\big) \quad \text{in } \Omega_h \\ &= -\Delta_h\big(\lambda\tfrac{4}{h^2}v_h + \lambda\tfrac{4}{hh_{min}}w_h\big) \quad \text{in } \Omega_h. \end{aligned}$$

Applying the maximum principle in Lemma 2.3 to this inequality above and using Lemmas 2.4 and 2.5, we induce that

$$u \ \leq \ \lambda\tfrac{4}{h^2}v_h + \lambda\tfrac{4}{hh_{min}}w_h \leq \tfrac{\lambda}{h^2}\big(4C_v + \tfrac{4h^3}{h_{min}}\big).$$

Consequently, we obtain $u \leq \lambda h^{-2}\big(4C_v + \frac{4h^3}{h_{min}}\big)$ in Ω_h so that $1 \leq \lambda h^{-2}\big(4C_v + \frac{4h^3}{h_{min}}\big)$ because $\max_{i=1,\ldots,K} u_i = 1$. Combining $\lambda \leq 2$, we obtain the bounds (8) for λ. If $h_{min} \geq h^3$, furthermore, then $C^{-1} := 4C_v + \frac{4h^3}{h_{min}} \leq 4C_v + 4$. In this case, $\lambda \geq Ch^2$, which completes the proof. □

Remark 3.4 *In [15], we showed that most domains with smooth boundary as well as rectangular domains satisfy $h_{min} = O(h^3)$. The domain Ω is called to have the general intersection property if the cumulative distribution function $p(\nu)$ defined by*

$$p(\nu) := |\{(x_i, y_j) \in \Omega_h : dist\,((x_i, y_j), \Gamma_h) \leq \nu\}| \tag{10}$$

is almost linear, i.e, $p(\nu) = O(h^{-2}\nu)$. Most domains have the general intersection property (see Table 1 for example and [15] for details). Note that when the cumulative distribution function $p(\nu)$ is almost linear, then for $\alpha > 2$, $p(h^\alpha) = O(h^{\alpha-2})$ and it means $h_{min} \geq h^\alpha$ for sufficiently small h.

Corollary 3.5 *If $h_{min} \geq h^3$, the condition number of the Jacobi preconditioned matrix is bounded by $O(h^{-2})$.*

The Jacobi case on Table 2 shows that the bound is tight. We observe that the Jacobi-preconditioning discretization (7) is preferred than the original discretization (2) in two senses. Its associated matrix has much smaller condition number ($O(h^{-2})$) than that of the original one ($O(\frac{1}{h\cdot h_{min}})$), which becomes $O(h^{-4})$ when $h_{min} \geq h^3$. This is due to the fact that the presence of grid nodes very near the boundary makes a malicious one in the original one. Also, the Gerschgorin circles of the matrix A are wide spread and, however, the Jacobi preconditioning collocates the circles sharing the same center. All the concentration of circles enhances the condition number of the matrix.

3.2 SGS and ILU preconditioning

In order to enhance the condition number, we consider preconditioning technique. The Jacobi and Symmetric Gauss-Seidal (SGS) preconditioners for A are $D^{-1}A$ and $(D+L)D^{-1}(D+U)$. Also the ILU and Modified ILU (MILU) preconditioners M for A are given as $M = (E+L)E^{-1}(E+U)$ where E is obtaining from the conditions

$$\begin{aligned} (E + LE^{-1}U)_{ii} &= D_{ii}, \quad i = 1, \ldots, K, \\ \sum_{j=1}^{K} (E + LE^{-1}U)_{ij} &= D_{ii}, \quad i = 1, \ldots, K. \end{aligned} \tag{11}$$

for ILU and MILU, respectively. Here C_{ij} denotes the ij entry element of a matrix C. We can show that the Jacobi preconditioning is invariant under SGS, ILU and MILU preconditioning. In the following, we show it only for SGS and the proofs for ILU and MILU are almost the same.

Theorem 3.6 (SGS+Jacobi = SGS) *Let A be the associated matrix of the Gibou et al. method, and let D be its Jacobi preconditioner and finally let M_A be its SGS preconditioner. Then*

$$M_A^{-1}A = M_{D^{-1}A}^{-1}(D^{-1}A)$$

where $M_{D^{-1}A}$ is the SGS preconditioner for $D^{-1}A$.

Proof Decompose A as $A = L + D + U$ where L, D, and U are the diagonal, the strict lower triangular, and the upper triangular parts of A, respectively. Then the SGS preconditioner M_A for A is given as $M_A = (D+L)D^{-1}(D+U)$. Likewise, since $D^{-1}A = D^{-1}L + I + D^{-1}U$, the SGS preconditioner $M_{D^{-1}A}$ for $D^{-1}A$ is given as $M_{D^{-1}A} = (I + D^{-1}L)(I + D^{-1}U)$. Then, using $D(I + D^{-1}L) = (D + L)$ and $(I + D^{-1}U) = D^{-1}(D+U)$, we have

$$DM_{D^{-1}A} = D(I + D^{-1}L)(I + D^{-1}U) = (D+L)(I + D^{-1}U) = (D+L)D^{-1}(D+U) = M_A,$$

and this shows the theorem. $\square$

The theorem above shows that the three preconditioners are actually applied on top of the application of Jacobi. Hence their effects are at least as good as Jacobi; Table 2 confirms this.

3.3 Modified ILU preconditioning

Now, we consider the MILU preconditioner of A in order to enhance the condition number. The MILU precondition M is given as $M = (E+L)E^{-1}(E+U)$ where E is obtained from the conditions (11). In this case, $M = (L+E)E^{-1}(U+E)$ is written as

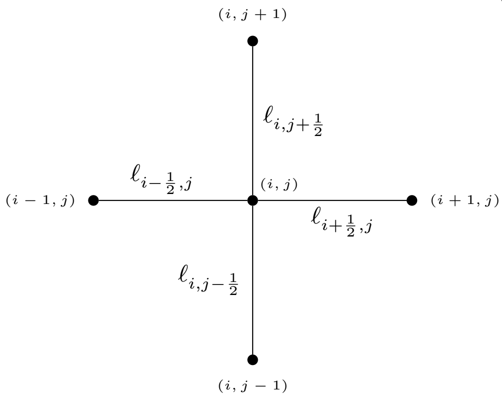
$$M = L + LE^{-1}L^T + L^T + E = A + R \quad (R := LE^{-1}L^T + E - D). \tag{12}$$

Let $e_{(i_k,j_k)}$ be the diagonal element of E corresponding to the node point $\mathbf{x}_k = (i_k h, j_k h)$ for $k = 1, \ldots, K$. From (11) and (12), we have

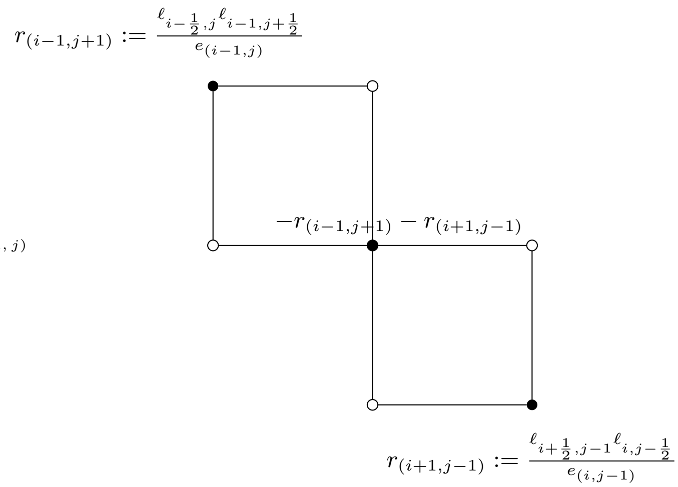
$$e_{(i_1,j_1)} = a_{1,1} \quad \text{and} \quad e_{(i_k,j_k)} = a_{k,k} - \frac{\ell_{i_k-\frac{1}{2},j_k}}{e_{(i_k-1,j_k)}}(\ell_{i_k-\frac{1}{2},j_k} + \ell_{i_k-1,j_k+\frac{1}{2}}) - \frac{\ell_{i_k,j_k-\frac{1}{2}}}{e_{(i_k,j_k-1)}}(\ell_{i_k,j_k-\frac{1}{2}} + \ell_{i_k+\frac{1}{2},j_k-1}) \tag{13}$$

and all the entries $e_{(i,j)}$ are determined recursively. Here $\ell_{i_k-\frac{1}{2},j}, \ell_{i_k-1,j_k+\frac{1}{2}}, \ell_{i_k,j_k-\frac{1}{2}}, \ell_{i_k+\frac{1}{2},j_k-1}$ are defined as

$$\ell_{i_k-\frac{1}{2},j_k} = \begin{cases} h^{-2}, & \text{if } \big((i_k-1)h, j_k h\big) \in \Omega_h \\ 0, & \text{otherwise} \end{cases}$$



(a) Matrix A



(b) Matrix R

Figure 2: Matrices A and $R = LE^{-1}L^T + E - D$

and

$$\ell_{i_k-1,j_k+\frac{1}{2}} = \begin{cases} h^{-2}, & \text{if } \big((i_k-1)h, j_kh\big), \big((i_k-1)h, (j_k+1)h\big) \in \Omega_h \\ 0, & \text{otherwise} \end{cases}$$

and $\ell_{i_k,j_k-\frac{1}{2}}, \ell_{i_k+\frac{1}{2},j_k-1}$ are defined for the points $\big(i_kh, (j_k-1)h\big), \big((i_k+1)h, (j_k-1)h\big)$ in the same way. In order to estimate the values of $e_{(i_k,j_k)}$, we need the following lemma.

Lemma 3.7 *For $k_1 = i_1 + j_1$, let $\{c_n\}_{n=k_1}^{\infty}$ be a sequence defined recursively as*

$$c_{k_1} = 4 \quad \textit{and} \quad c_{n+1} = 4 - \frac{4}{c_n}, \quad n \geq k_1. \tag{14}$$

Then, we have

$$c_n \geq 2 + \frac{2}{n}, \quad \textit{for } n \geq k_1.$$

Proof Let $\{c_n\}_{n=k_1}^{\infty}$ be the sequence defined as (14). The lemma is shown by the mathematical induction. Assume that $c_n \geq 2 + 2/n$, for $n = k_1, \ldots, k$. Then

$$c_{k+1} = 4 - \frac{4}{c_k} \geq 4 - \frac{2k}{k+1} = 2 + \frac{2}{k+1},$$

and this proves the lemma. $\square$

Theorem 3.8 *Let $M = (L+E)E^{-1}(U+E)$ be the MILU preconditioner for A. Then, for every diagonal element $e_{(i_k,j_k)}$ of E corresponding to the node $(i_kh, j_kh) \in \Omega_h$, we have*

$$h^2 e_{(i_k,j_k)} \geq 2 + \frac{2}{i_k + j_k} \quad \textit{for } k = 1, \ldots, K$$

and, therefore,

$$\|e_{(i_k,j_k)}\|_\infty \geq 2h^{-2} + \frac{2}{a+b}h^{-1}.$$

Proof First, we shall show that for the sequence $\{c_n\}_{n=k_1}^{\infty}$ defined in (14), we have

$$h^2 e_{(i_k,j_k)} \geq c_{i_k+j_k}, \quad k = 1, \dots, K. \tag{15}$$

Since $h^2 e_{(i_1,j_1)} \geq c_{k_1} = 4$, the inequality holds for $k = 1$. Now, we assume that there exists $\tau \geq 2$ such that for all $k = 1, \dots, \tau - 1$

$$h^2 e_{(i_k,j_k)} \geq c_{i_k+j_k}.$$

From (13), it is not difficult to see that all the entries $e_{(i_k,j_k)}$ are positive and we obtain

$$\begin{aligned}
h^2 e_{(i_\tau,j_\tau)} &= h^2 a_{\tau,\tau} - h^2 \frac{\ell_{i_\tau-\frac{1}{2},j_\tau}}{e_{(i_\tau-1,j_\tau)}}(\ell_{i_\tau-\frac{1}{2},j_\tau} + \ell_{i_\tau-1,j_\tau+\frac{1}{2}}) - h^2 \frac{\ell_{i_\tau,j_\tau-\frac{1}{2}}}{e_{(i_\tau,j_\tau-1)}}(\ell_{i_\tau,j_\tau-\frac{1}{2}} + \ell_{i_\tau+\frac{1}{2},j_\tau-1}) \\
&\geq 4 - \frac{2}{h^2 e_{(i_\tau-1,j_\tau)}} - \frac{2}{h^2 e_{(i_\tau,j_\tau-1)}} \\
&\geq 4 - \frac{4}{c_{i_\tau+j_\tau-1}} = c_{i_\tau+j_\tau}.
\end{aligned}$$

Thus, the mathematical induction verifies the relations (15). Applying the relations (15) to Lemma 3.7 gives

$$h^2 e_{(i_k,j_k)} \geq c_{i_k+j_k} \geq 2 + \frac{2}{i_k + j_k}, \quad k = 1, \dots, K,$$

and this proves the first claim of the theorem. Using the lower bounds and the relations $i_k \leq N \leq a/h$ and $j_k \leq M \leq b/h$, we also obtain

$$\|h^2 e_{(i_k,j_k)}\|_\infty \geq 2 + \frac{2}{N+M} \geq 2 + \frac{2}{a+b}h,$$

which completes the proof. $\square$

Now, we are ready to estimate the condition number of the MILU preconditioned matrix $M^{-1}A$. We can show that all the eigenvalues of $M^{-1}A$ are real and positive. Moreover, the minimum and maximum eigenvalues of $M^{-1}A$ are given as

$$\lambda_{min} = \min_{v \in \mathbb{R}^K} \frac{\langle Av, v \rangle}{\langle Mv, v \rangle} \quad \text{and} \quad \lambda_{max} = \max_{v \in \mathbb{R}^K} \frac{\langle Av, v \rangle}{\langle Mv, v \rangle} \tag{16}$$

and $\langle Av, v \rangle / \langle Mv, v \rangle$ is written in the form

$$\frac{\langle Av, v \rangle}{\langle Mv, v \rangle} = \frac{1}{1 + \langle Rv, v \rangle / \langle Av, v \rangle} \tag{17}$$

for the matrix $R = M - A$ (see (12) and (b) of Fig. 2 for its entries). For $v \neq 0$, we can write

$$\begin{aligned}
\langle Av, v \rangle &= -\sum_r \sum_{s>r} a_{rs}(v_r - v_s)^2 + \sum_r v_r^2 \sum_s a_{rs} \\
&\geq \sum_k (\ell_{i_k+\frac{1}{2},j_k}(v_{(i_k,j_k)} - v_{(i_k+1,j_k)})^2 + \ell_{i_k,j_k+\frac{1}{2}}(v_{(i_k,j_k)} - v_{(i_k,j_k+1)})^2)
\end{aligned} \tag{18}$$

Applying the zero row sum property of R and Theorem 3.8, and using the inequality $(x+y)^2 \leq$

$2(x-z)^2+2(y-z)^2$, we also have

$$\begin{aligned}
-\langle Rv, v\rangle &= \sum_r \sum_{s>r} r_{rs}(v_r - v_s)^2 = \sum_k r_{(i_k-1,j_k+1)}(v_{(i_k,j_k)} - v_{(i_k-1,j_k+1)})^2 \\
&= \sum_{k=2}^{K} \frac{\ell_{i_k-\frac{1}{2},j_k}\ell_{i_k-1,j_k+\frac{1}{2}}}{e_{(i_k-1,j_k)}}(v_{(i_k,j_k)} - v_{(i_k-1,j_k+1)})^2 \\
&\le \frac{a+b}{(a+b)+h}\sum_{k=1}^{K}(\ell_{i_k+\frac{1}{2},j_k}(v_{(i_k,j_k)} - v_{(i_k+1,j_k)})^2 + \ell_{i_k,j_k+\frac{1}{2}}(v_{(i_k,j_k)} - v_{(i_k,j_k+1)})^2) \\
&\le \frac{a+b}{(a+b)+h}\langle Av, v\rangle.
\end{aligned}$$

Thus, we obtain the inequalities

$$0 \le \frac{-\langle Rv, v\rangle}{\langle Av, v\rangle} \le \frac{a+b}{(a+b)+h}. \tag{19}$$

In summary, we have the following.

Theorem 3.9 *Let λ be an eigenvalue of the MILU preconditioned matrix, then*

$$\lambda_{min} = 1 \le \lambda \le 1 + \frac{a+b}{h} = O(h^{-1}).$$

Proof Let λ be an eigenvalue of the MILU preconditioned matrix $M^{-1}A$. Using the symmetry of A and $E \ge 0$, we can show that $\lambda > 0$. From (19), we have that

$$\frac{h}{a+b+h} \le 1 + \frac{\langle Rv, v\rangle}{\langle Av, v\rangle} \le 1, \quad \forall v \ne 0,$$

and applying these inequalities above into (16) and (17) gives

$$1 \le \lambda \le 1 + \frac{a+b}{h}.$$

On the other hand, from (11) and (12), A and M have the same row sums, that is

$$Ax = Mx, \quad x = (1, \ldots, 1)^T \in \mathbb{R}^K,$$

and this shows $\lambda_{min} = 1$, which completes the proof. □

Corollary 3.10 *The ratio of the maximum and minimum eigenvalues of the MILU preconditioned matrix is bounded by $O\left(h^{-1}\right)$.*

We note that when the domain Ω is a rectangle, the Gibou *et al.* scheme becomes the standard five point scheme, and Theorem 3.9 and its corollary cover the results for the standard five point scheme on a rectangular domain ([16]).

Remark 3.11 *The excellence of MILU over the others was actually observed and proved [6, 5] for the Poisson solver in rectangular domains. For the ease of proof, however, he added a diagonal perturbation to turn the diagonal dominance of the linear system into a strictly diagonal dominance. Actually it was Gustafsson's conjecture [1] to prove the number $O(h^{-1})$ for the unperturbed MILU, and the conjecture was proved [11].*

Our proof of the estimate $O(h^{-1})$ for MILU was performed without any artificial perturbation. Though unnecessary, the diagonal perturbation has been still used as a cliche; e.g., see the mixture of 96% MILU and 4% ILU in [2]. Note that 100% MILU is enough to achieve the success $O(h^{-1})$.

grid	Unpreconditioned matrix			
	λ_{min}	λ_{max}	$h_{min} = h^{\alpha}$	$\left(\frac{8}{h \cdot h_{min}}\right)/\lambda_{max}$
20^2	5.74	3.18×10^2	$5.04 \times 10^{-2} = h^{1.62}$	3.17
40^2	5.77	1.12×10^4	$1.54 \times 10^{-3} = h^{2.53}$	6.04
80^2	5.78	5.58×10^4	$7.03 \times 10^{-4} = h^{2.21}$	5.37
160^2	5.78	8.03×10^5	$0.92 \times 10^{-4} = h^{2.34}$	5.76
320^2	5.78	1.12×10^7	$0.13 \times 10^{-4} = h^{2.41}$	5.78

Table 1: Eigenvalues of the unpreconditioned matrix: the results obey the estimate of Theorem 3.1, $0 < \frac{1}{C} \leq \lambda \leq \frac{8}{h \cdot h_{min}}$. Also note that $h_{min} \geq h^3$ which supports the general intersection property mentioned in Remark 3.4.

grid	Jacobi preconditioned				SGS preconditioned			
	λ_{max}	ratio	λ_{min}	ratio	λ_{max}	ratio	λ_{min}	ratio
20^2	1.964		35.73×10^{-3}		1.000		129.681×10^{-3}	
40^2	1.992	1.013	8.540×10^{-3}	0.239	1.000	1.000	33.326×10^{-3}	0.257
80^2	1.998	1.003	2.08×10^{-3}	0.244	1.000	1.000	8.286×10^{-3}	0.249
160^2	2.000	1.000	0.515×10^{-3}	0.247	1.000	1.000	2.056×10^{-3}	0.248
320^2	2.000	1.000	0.128×10^{-3}	0.249	1.000	1.000	0.511×10^{-3}	0.249

grid	ILU preconditioned				MILU preconditioned			
	λ_{max}	ratio	λ_{min}	ratio	λ_{max}	ratio	λ_{min}	ratio
20^2	1.190		208.573×10^{-3}		3.284		1.000	
40^2	1.202	1.011	56.029×10^{-3}	0.267	6.648	2.024	1.000	1.000
80^2	1.206	1.003	14.090×10^{-3}	0.251	13.478	2.027	1.000	1.000
160^2	1.206	1.000	3.506×10^{-3}	0.248	27.279	2.024	1.000	1.000
320^2	1.207	1.001	0.873×10^{-3}	0.249	55.075	2.019	1.000	1.000

Table 2: Eigenvalues of the preconditioned matrices: the results of Jacobi, SGS, and ILU tightly obey the estimate $O(h^{-2}) < \lambda < O(1)$, and MILU excels the others with the smaller ratio $O(h^{-1})$.

3.4 Numerical test

Let $\Omega \subset \left[-\frac{3}{2}, \frac{3}{2}\right]^2$ be a circle of center $(0,0)$ and radius 1. Choose $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ so that $u\,(x,y) = \frac{y}{(x+2)^2+y^2}$ becomes the exact solution of the Poisson problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \Gamma. \end{aligned}$$

Let A be the matrix associated with the Gibou *et al.* discretization (2) of the above problem. Table 1 verifies Theorem 3.1 that $\lambda_{min}\,(A) = O\,(1)$ and $\lambda_{max}\,(A) = O\left(\frac{1}{h \cdot h_{min}}\right)$. Table 2 verifies Theorems 3.3 and 3.9 that $\lambda_{min}\left(M^{-1}A\right) = O\left(h^2\right)$ and $\lambda_{max}\left(M^{-1}A\right) = O\,(1)$ in the cases of Jacobi, SGS, and ILU preconditioners. In the table, MILU shows more improvement than the others in eigenvalue clustering as $\lambda_{min}\left(M^{-1}A\right) = O\,(1)$ and $\lambda_{max}\left(M^{-1}A\right) = O\left(\frac{1}{h}\right)$.

4 Conclusion

We have introduced two analyses for the finite difference method [4]. One analysis proves that its solution is second order accurate to the analytic solution, and the other estimates the condition number of its linear systems with and without preconditioners. The unpreconditioned linear system has a very large condition number $O\left(1/\left(h \cdot h_{min}\right)\right)$, required to be preconditioned. Our estimates put into consideration many of very popular preconditioners; Jacobi, ILU, SGS, and MILU. Our estimate definitely suggests MILU preconditioning among them, since it reduces the condition number from $O\left(1/\left(h \cdot h_{min}\right)\right)$ to $O(h^{-1})$, while the others to $O(h^{-2})$.

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