CONVERGENCE ANALYSIS ON THE GIBOU-MIN METHOD FOR THE HODGE PROJECTION*

GANGJOON YOON†, JAE-HYUN PARK‡, AND CHOHONG MIN§

Abstract. The Hodge projection of a vector field is the divergence-free component of its Helmholtz decomposition. In a bounded domain, a boundary condition needs to be supplied to the decomposition. The decomposition with the non-penetration boundary condition is equivalent to solving the Poisson equation with the Neumann boundary condition. The Gibou-Min method is an application of the Poisson solver by Purvis and Burkhalter to the decomposition.

In the decomposition by the Gibou-Min method, an important L^2-orthogonality holds between the gradient field and the solenoidal field, which is similar to the continuous Hodge decomposition.

Using the orthogonality, we present a novel analysis which shows that the convergence order is 1.5 in the L^2-norm for approximating the divergence-free vector field. Numerical results are presented to validate our analyses.

Key words. Hodge projection, finite volume method, Poisson equation, Gibou-Min.

Subject classification. 65N06.

1. Introduction

The Helmholtz decomposition theorem [8] states that any smooth vector field U^* can be decomposed into the sum of a gradient field ∇p and a divergence-free vector field U. The decomposition is unique and orthogonal in L^2. The Hodge projection of a vector field is defined as the divergence-free component in its Helmholtz decomposition.

One of the main applications of the Hodge projection is the incompressible fluid flow, whose phenomenon is represented by the Navier-Stokes equations. Consisting of the conservation equation of momentum and the state equation of divergence-free condition, the equations can be described by a convection-diffusion equation with the Hodge projection applied at every moment. Chorin's seminal approximation [3] for the fluid flow first solves the convection-diffusion equation in a usual manner, and then applies the Hodge projection. Other successful fluid solvers such as Kim and Moin's [9], Bell et al.'s [2], Gauge method [4] are in the same direction as that of Chorin's.

The Helmholtz decomposition $U^* = U + \nabla p$ in a domain Ω can be implemented through the Poisson equation $-\Delta p = -\nabla \cdot U^*$. In a bounded domain, the equation needs to be supplied with boundary condition. There are two types of fluid boundary conditions. One is the non-penetration boundary condition, $U \cdot n = 0$ on $\Gamma = \partial\Omega$, and the other is the free boundary condition, $p = \sigma\kappa$ on Γ [11]. The free boundary condition is, in other words, the Dirichlet boundary condition of the Poisson equation, and the non-penetration boundary condition corresponds to the Neumann boundary condition, $\frac{\partial p}{\partial n} = U^* \cdot n$ on Γ.

A classical finite difference method for solving the Poisson equation with the Dirichlet boundary condition is the Shortley-Weller method. It has been long known to have a second order accurate solution, but it is recent to have mathematical analyses

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†Institute of Mathematical Sciences, Ewha Womans University, Seoul, Korea 120-750

‡Department of Mathematics, Kunsan National University, Kunsan, Korea 573-701

§Corresponding author, Department of Mathematics, Ewha Womans University, Seoul, Korea 120-750, (chohong@ewha.ac.kr)

for the convergence of its gradient. Le et al. showed the second order accuracy in rectangular domains in 2004 and the order 1.5 in polygonal domains in 2007, and Yoon and Min showed the second order accuracy in any smooth domains [13]. One weakness of the Shortley-Weller method is that its associated linear system is not symmetric. Gibou et al [5] introduced a symmetric modification of the method while keeping the same second order accuracy for the solution.

A standard finite difference/volume method for the Poisson equation with the Neumann boundary condition was introduced by Purvis and Burkhalter [12]. Though implemented in uniform grid, the method can handle arbitrarily shaped domains. It is a simple modification of the standard five-point finite difference method, and it constitutes a five-banded sparse linear system that is diagonally dominant, symmetric and positive semi-definite. Due to these nice properties, the linear system can be efficiently solved by the Conjugate Gradient method with various efficient ILU preconditioners.

The Gibou-Min method [10] is an application of the Purvis-Burkhalter method on the Hodge decomposition. In implementing the Hodge decomposition, the Neumann boundary condition takes the divergence form $\frac{\partial p}{\partial n} = \nabla \cdot U^*$, which delivers an L^2 orthogonality of the decomposition. The Gibou-Min method was applied to fluid-solid interaction [6].

We briefly review the Gibou-Min method in section 2 where the discrete gradient and divergence operators are introduced. In section 3, we show the discrete integration-by-parts which leads to a discrete orthogonal decomposition. After decomposing the consistency of the method, we prove an order of accuracy of 1.5 in the L^2-norm for approximating the divergence-free vector field U of the Hodge projection. Numerical results are presented in section 4 to validate our analyses. Our convergence analysis is novel to our best search.

2. Numerical method

In this section, we briefly review the Gibou-Min method [10] for the Hodge decomposition with the non-penetration boundary condition. Given a vector field U^* in a bounded and connected domain Ω, the following Poisson equation is solved for scalar p with the Neumann boundary condition.

$$\begin{cases} -\Delta p = -\nabla \cdot U^* & \text{in } \Omega, \\ \frac{\partial p}{\partial n} = U^* \cdot n & \text{on } \Gamma := \partial\Omega. \end{cases} \tag{2.1}$$

Then a vector field U, which is defined as $U = U^* - \nabla p$, is the desired Hodge projection of U^* that satisfies the divergence-free condition $\nabla \cdot U = 0$ in Ω, and the non-penetration boundary condition $U \cdot n = 0$ on Γ. The Gibou-Min method samples the vector fields and scalar field on the Marker-and-Cell (MAC) staggered grid [7]. For simplicity, we consider the 2D case in this section and the extension to 3D is straightforward.

Let $h\mathbb{Z}^2$ denote the uniform grid in $\mathbb{R}^2$ with step size h. For each grid node $(x_i, y_j) \in h\mathbb{Z}^2$, C_{ij} denotes the rectangular control volume centered at the node, and its four edges are denoted by $E_{i\pm\frac{1}{2},j}$ and $E_{ij\pm\frac{1}{2}}$ as follows.

$$\begin{aligned} C_{ij} &:= [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}], \\ E_{i\pm\frac{1}{2}j} &:= \quad x_{i\pm\frac{1}{2}} \quad \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}], \\ E_{ij\pm\frac{1}{2}} &:= [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times \quad y_{j\pm\frac{1}{2}}. \end{aligned}$$

Based on the MAC configuration, we define the node set and the edge sets.

DEFINITION 2.1 (Node set and edge sets). *$\Omega^h := \left\{(x_i,y_j)\in h\mathbb{Z}^2 | C_{ij}\cap\Omega\neq\emptyset\right\}$ is the set of nodes whose control volumes intersecting the domain. In the same way, defined are edge sets $E_x^h := \left\{(x_{i+\frac{1}{2}},y_j) | E_{i+\frac{1}{2},j}\cap\Omega\neq\emptyset\right\}$ and $E_y^h := \left\{(x_i,y_{j+\frac{1}{2}}) | E_{i,j+\frac{1}{2}}\cap\Omega\neq\emptyset\right\}$, and then $E^h := E_x^h\cup E_y^h$.*

By the standard central finite differences, a discrete gradient operator is defined.

DEFINITION 2.2 (Discrete gradient). *Given $p:\Omega^h\to\mathbb{R}$, its gradient $Gp:E^h\to\mathbb{R}$ is defined as*

$$
\begin{aligned}
(G^x p)_{i+\frac{1}{2},j} &= \frac{p_{i+1,j}-p_{ij}}{h},\\
(G^y p)_{i,j+\frac{1}{2}} &= \frac{p_{ij+1}-p_{ij}}{h}.
\end{aligned}
$$

Whenever $E_{i+\frac{1}{2},j}\cap\Omega\neq\emptyset$, $C_{ij}\cap\Omega\neq\emptyset$ and $C_{i+1,j}\cap\Omega\neq\emptyset$, since $E_{i+\frac{1}{2},j}\subset C_{ij},C_{i+1,j}$. Hence the above definition is well posed for $G^x p$, and so is for $G^y p$. Discrete gradient was simply defined by the finite differences, however discrete divergence can not be defined so. For each $(x_i,y_j)\in\Omega^h$, its four neighboring edges may not be in E^h, since $C_{ij}\cap\Omega\neq\emptyset$ neither imply $E_{i\pm\frac{1}{2},j}\cap\Omega\neq\emptyset$ nor $E_{i,j\pm\frac{1}{2}}\cap\Omega\neq\emptyset$. A proper definition comes from the following identity.

$$
\begin{aligned}
\int\limits_{C_{ij}\cap\Omega}\nabla\cdot U\,dx &= \int\limits_{\partial(C_{ij}\cap\Omega)} U\cdot\vec{n}\,ds\\
0 &= \int\limits_{\partial C_{ij}\cap\Omega} U\cdot\vec{n}\,ds + \int\limits_{C_{ij}\cap\Gamma} U\cdot\vec{n}\,ds
\end{aligned}
\tag{2.2}
$$

With the non-penetration boundary condition $U\cdot\vec{n}=0$, the identity represents an integral value of the divergence by a line integral over the fraction of edges. To measure the fraction, the following Heaviside functions are defined on the edge set. We note that the Heaviside function is sampled only on edge, not in cells. Thus, $C_{ij}\cap\Omega$ is not needed in Gibou-Min method, but will be needed in Purvis-Burkhalter's [12].

DEFINITION 2.3 (Heaviside function). *For each edge,*

$$
H_{i+\frac{1}{2},j} = \frac{length\left(E_{i+\frac{1}{2},j}\cap\Omega\right)}{length\left(E_{i+\frac{1}{2},j}\right)},\ \textit{and}\ H_{i,j+\frac{1}{2}} = \frac{length\left(E_{i,j+\frac{1}{2}}\cap\Omega\right)}{length\left(E_{i,j+\frac{1}{2}}\right)}.
$$

Note that $H_{i+\frac{1}{2},j},\ H_{i,j+\frac{1}{2}}\in[0,1]$. Its value 1 implies that the edge is totally inside the domain, and value 0 implies completely outside. One may approximate the Heaviside function in different ways: Batty and Bridson [1] approximated it by volume fraction instead of edge fraction. Between the two approximations, the edge fraction, which the Gibou-Min method uses, shows much better convergence [10]. Using the Heaviside function, now we define discrete divergence operator.

DEFINITION 2.4 (Discrete divergence). *Given $U=(u,v):E^h\to\mathbb{R}$, its discrete divergence $DU:\Omega^h\to\mathbb{R}$ is defined as*

$$
\begin{aligned}
(DU)_{ij} = &\left(u_{i+\frac{1}{2},j}H_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}H_{i-\frac{1}{2},j}\right)\cdot h\\
&+\left(v_{i,j+\frac{1}{2}}H_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}H_{i,j-\frac{1}{2}}\right)\cdot h.
\end{aligned}
$$

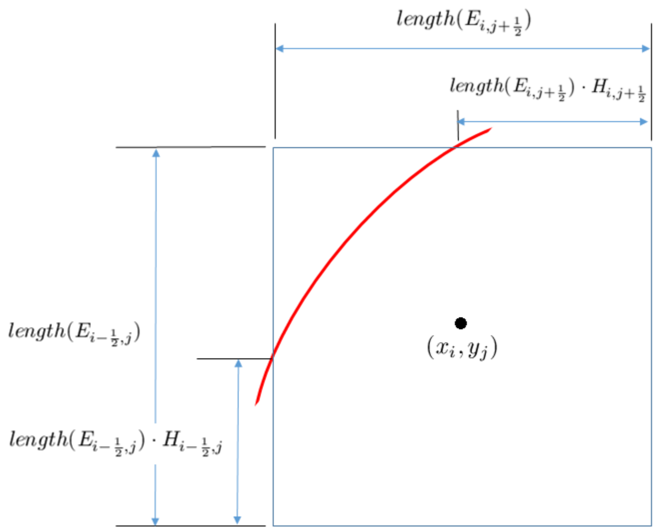

FIG. 2.1. *The rectangular control volume C_{ij} has four edges $E_{i\pm\frac{1}{2},j}$ and $E_{i,j\pm\frac{1}{2}}$. The Heaviside function is defined to be $H_{i\pm\frac{1}{2},j}, H_{i,j\pm\frac{1}{2}} \in [0,1]$ on the edges.*

Note that the calculation of the discrete divergence involves the vector field only in E^h. The edges not in E^h, whose Heaviside function values are zero, are ignored in the calculation.

Given a vector field $U^*: E^h \to \mathbb{R}$, the Gibou-Min method computes a vector field $U^h: E^h \to \mathbb{R}$ and a scalar field $p^h: \Omega^h \to \mathbb{R}$ such that $DU^h = 0$ in Ω^h and $U^* = U^h + Gp^h$ in Ω^h. Substituting U^h with $U^* - Gp^h$ in $DU^h = 0$, we have the equation for p^h,

$$-DGp^h = -DU^* \text{ in } \Omega^h. \tag{2.3}$$

After p^h is obtained by solving the above linear system, the solenoidal vector field $U^h = U^* - Gp^h$ is calculated.

3. Convergence analysis

In this section, we perform convergence analysis for the Gibou-Min method. Let $L^h := DG$ denote its associated linear operator, then it maps a discrete function $p^h: \Omega^h \to \mathbb{R}$ to another function $L^h p^h: \Omega^h \to \mathbb{R}$ such that

$$\begin{aligned}\left(L^h p^h\right)_{ij} = \quad & H_{i+\frac{1}{2},j}\left(p^h_{i+1,j} - p^h_{ij}\right) - H_{i-\frac{1}{2},j}\left(p^h_{ij} - p^h_{i-1j}\right) \\ & + H_{i,j+\frac{1}{2}}\left(p^h_{i,j+1} - p^h_{ij}\right) - H_{i,j-\frac{1}{2}}\left(p^h_{ij} - p^h_{ij-1}\right),\end{aligned} \tag{3.1}$$

for each $(x_i, y_j) \in \Omega^h$. Then we first show that the equation

$$-L^h p^h = -DU^* \tag{3.2}$$

has a unique solution p^h satisfying $\sum_{i,j} \left(p^h\right)_{ij} = 0$.

LEMMA 3.1. *$Ker\left(L^h\right) = span\{1_{\Omega^h}\}$*

Proof. Let $p^h:\Omega^h\to\mathbb{R}$ be a discrete function with $L^hp^h\equiv 0$. Since Ω^h is a finite set, $\max_{\Omega^h}p^h=p^h_{i^*j^*}$ for some $(x_{i^*},y_{j^*})\in\Omega^h$. At the location, since $p^h_{i^*j^*}$ is the maximum, we have the following inequality.

$$\begin{aligned} &H_{i+\frac{1}{2},j}\left(p^h_{i^*+1,j^*}-p^h_{i^*j^*}\right)-H_{i-\frac{1}{2},j}\left(p^h_{i^*,j^*}-p^h_{i^*-1,j^*}\right)\\ &+H_{ij+\frac{1}{2}}\left(p^h_{i^*,j^*+1}-p^h_{i^*j^*}\right)-H_{ij-\frac{1}{2}}\left(p^h_{i^*,j^*}-p^h_{i^*,j^*-1}\right)\ \le 0. \end{aligned}$$

To have the equality $\left(L^hp^h\right)_{i^*j^*}=0$, $p^h_{i^*j^*}$should be equal to all of its neighbors. Now the maximum is propagated to all of its neighbors. Recursively applying the same idea to each neighborhood, the maximum should be propagated to the whole region. Hence p^h is globally constant. The other direction is trivial. □

THEOREM 3.2 (Existence and uniqueness). *For a vector field $U^*:E^h\to\mathbb{R}$, $-L^hp^h=-DU^*$ has a unique solution $p^h\in\{1_{\Omega^h}\}^\perp$.*

Proof. Since L^h is symmetric, the range of L^h is $\{1_{\Omega^h}\}^\perp$. And since the function DU^* satisfies a compatibility condition $\sum_{i,j}(DU^*)_{ij}=0$, namely, DU^* is in the range $\{1_{\Omega^h}\}^\perp$, there exists an unique solution $p^h\in\{1_{\Omega^h}\}^\perp$. □

3.1. Discrete integration-by-parts

Two operators $G:\mathbb{R}^{\Omega^h}\to\mathbb{R}^{E^h}$ and $D:\mathbb{R}^{E^h}\to\mathbb{R}^{\Omega^h}$ satisfy the following discrete integration-by-parts.

DEFINITION 3.3 (Inner product between vector fields). *Given two vector fields $U,V:E^h\to\mathbb{R}$, their inner product is defined as*

$$\langle U,V\rangle:=\sum_{i,j}u^1_{i+\frac{1}{2},j}v^1_{i+\frac{1}{2},j}H_{i+\frac{1}{2},j}h^2+\sum_{i,j}u^2_{i,j+\frac{1}{2}}v^2_{i,j+\frac{1}{2}}H_{i,j+\frac{1}{2}}h^2$$

where $U=(u^1,u^2)$ and $V=(v^1,v^2)$.

DEFINITION 3.4 (Inner product between scalar fields). *Given two discrete functions $p^1,p^2:\Omega^h\to\mathbb{R}$, their inner product is defined as*

$$\left\langle p^1,p^2\right\rangle:=\sum_{i,j}p^1_{i,j}p^2_{i,j}h^2.$$

With the two inner products defined, the following lemma shows that G is the adjoint operator of $-\frac{1}{h^2}D$.

LEMMA 3.5 (Integration-by-parts). *For any function $p:\Omega^h\to\mathbb{R}$ and any vector field $U:E^h\to\mathbb{R}$, we have*

$$\langle Gp,U\rangle=-\left\langle p,\frac{1}{h^2}DU\right\rangle.$$

Proof. Let a vector field $U:E^h\to\mathbb{R}$ be given by $U=(u,v)$. Then it follows from the definition of inner product between vector fields that

$$\begin{aligned} \langle Gp,U\rangle&=\sum_{i,j}(Gp)_{i+\frac{1}{2},j}u_{i+\frac{1}{2},j}H_{i+\frac{1}{2},j}h^2+\sum_{i,j}(Gp)_{i,j+\frac{1}{2}}v_{i,j+\frac{1}{2}}H_{i,j+\frac{1}{2}}h^2\\ &=\sum_{i,j}\frac{p_{i+1,j}-p_{i,j}}{h}u_{i+\frac{1}{2},j}H_{i+\frac{1}{2},j}h^2+\sum_{i,j}\frac{p_{i,j+1}-p_{i,j}}{h}v_{i,j+\frac{1}{2}}H_{i,j+\frac{1}{2}}h^2\\ &=-\sum_{i,j}p_{ij}\left(u_{i+\frac{1}{2},j}H_{i+\frac{1}{2},j}-u_{i-\frac{1}{2},j}H_{i-\frac{1}{2},j}+v_{i,j+\frac{1}{2}}H_{i,j+\frac{1}{2}}-v_{i,j-\frac{1}{2}}H_{i,j-\frac{1}{2}}\right)h\\ &=-\left\langle p,\frac{1}{h^2}DU\right\rangle. \end{aligned}$$

□

The integration-by-parts leads to the following orthogonality theorem.

THEOREM 3.6 (L^2-orthogonality). *Given a vector field* $U^*: E^h \to \mathbb{R}$, *there exists a unique* $p^h \in \{1_{\Omega^h}\}^\perp$ *such that* $DGp^h = DU^*$. *Therefore, the decomposition*

$$U^* = U^h + Gp^h \quad \text{with } DU^h = 0$$

is unique. Furthermore, the decomposition is orthogonal, i.e. $\langle U^h, Gp^h \rangle = 0$.

Proof. The uniqueness of the decomposition is straightforwardly proved by the uniqueness of p^h. Moreover, since p^h satisfies $DGp = DU^*$, we have

$$0 = \langle DU^h, p^h \rangle = -h^2 \langle U^h, Gp^h \rangle.$$

□

3.2. Error estimate

Throughout this section, let $p: \Omega \to \mathbb{R}$ and $U = (u,v): \Omega \to \mathbb{R}^2$ denote the analytic solution of the Helmholtz decomposition $U^* = U + \nabla p$ for the given vector field $U^* = (u^*, v^*): \Omega \to \mathbb{R}^2$. Also let $p^h: \Omega^h \to \mathbb{R}$ and $U^h: E^h \to \mathbb{R}$ denote the numerical solution for the Gibou-Min method for the given $U^*: E^h \to \mathbb{R}$.

DEFINITION 3.7 (Convergence error). *The convergence error* $e^h: \Omega_h \to \mathbb{R}$ *is defined as* $e^h := p - p^h$.

DEFINITION 3.8 (Consistency). *The consistency* $c^h: \Omega_h \to \mathbb{R}$ *is defined as* $c^h = L^h p - L^h p^h$.

LEMMA 3.9 (Decomposition of consistency). $c^h = \frac{1}{h} Dd^h$, *where* $d^h: E^h \to \mathbb{R}$ *is defined as*

$$\begin{aligned} d^h_{i+\frac{1}{2},j} &:= \frac{1}{H_{i+\frac{1}{2},j}} \int\limits_{E_{i+\frac{1}{2}j} \cap \Omega} \left[\frac{p_{i+1,j} - p_{ij}}{h} - u^*(x_{i+\frac{1}{2}}, y_j) \right] dy \\ &+ \frac{1}{H_{i+\frac{1}{2},j}} \int\limits_{E_{i+\frac{1}{2}j} \cap \Omega} \left[u^*(x_{i+\frac{1}{2}}, y) - \frac{\partial p}{\partial x} \left(x_{i+\frac{1}{2}}, y \right) \right] dy, \end{aligned}$$

and $d^h_{i,j+\frac{1}{2}}$ *is defined in the same manner.*

Proof. Splitting the constant part and variable part in the integral, we have

$$\begin{aligned} H_{i+\frac{1}{2},j} d^h_{i+\frac{1}{2},j} = H_{i+\frac{1}{2},j} (p_{i+1,j} - p_{ij}) - H_{i+\frac{1}{2},j} h \cdot u^*_{i+\frac{1}{2},j} \\ + \int\limits_{E_{i+\frac{1}{2}j} \cap \Omega} (U^* - \nabla p) \cdot n \, dy. \end{aligned}$$

From the definition of divergence,

$$\begin{aligned} \frac{1}{h} \left(Dd^h \right)_{ij} = H_{i+\frac{1}{2},j} d^h_{i+\frac{1}{2},j} - H_{i-\frac{1}{2},j} d^h_{i-\frac{1}{2},j} \\ + H_{i,j+\frac{1}{2}} d^h_{i,j+\frac{1}{2}} - H_{i,j-\frac{1}{2}} d^h_{i,j-\frac{1}{2}}. \end{aligned}$$

Using the split, we have

$$\frac{1}{h}\left(Dd^h\right)_{ij} = H_{i+\frac{1}{2},j}\left(p_{i+1,j}-p_{ij}\right) - H_{i-\frac{1}{2},j}\left(p_{ij}-p_{i-1j}\right)+\cdots$$
$$-h\cdot H_{i+\frac{1}{2},j}u^*_{i+\frac{1}{2},j} + h\cdot H_{i-\frac{1}{2},j}u^*_{i-\frac{1}{2},j}+\cdots$$
$$+\int_{\partial C_{ij}\cap\Omega}(U^*-\nabla p)\cdot n\,ds.$$

Since $U=U^*-\nabla p$ satisfies $U\cdot n=0$ on Γ and $\nabla\cdot U=0$ in Ω, we have

$$0=\int_{C_{ij}\cap\Omega}\nabla\cdot U\,dx=\int_{\partial C_{ij}\cap\Omega}U\cdot n\,ds+\int_{C_{ij}\cap\Gamma}U\cdot n\,ds=\int_{\partial C_{ij}\cap\Omega}U\cdot n\,ds.$$

Thus,

$$\frac{1}{h}\left(Dd^h\right)_{ij}=\left(L^h p\right)_{ij}-\left(DU^*\right)_{ij}+0=\left(L^h p\right)_{ij}-\left(L^h p^h\right)_{ij}.$$

□

Definition 3.10 (Big-oh notation). *For a given quantity v_h depending on step size h, we write*

$$v_h=O(h^\alpha)$$

for some real number $\alpha\geq 0$, provided there exist two constants $C,D>0$ such that

$$|v_h|\leq Ch^\alpha$$

for all $h<D$.

Now we estimate the size of consistency in the decomposition.

Lemma 3.11. *For each i and j,*

$$d^h_{i+\frac{1}{2},j}=\begin{cases} O(h^3) \text{ if } H_{i+\frac{1}{2},j}=1,\\ O(h^2) \text{ if } 0<H_{i+\frac{1}{2},j}<1,\end{cases}\tag{3.3}$$

and the same result holds for $d^h_{i,j+\frac{1}{2}}$.

Proof. Using Taylor series expansions for $\frac{p_{i+1,j}-p_{ij}}{h}-\frac{\partial p}{\partial x}$ and for $u^*\left(x_{i+\frac{1}{2}},y\right)-u^*\left(x_{i+\frac{1}{2}},y_j\right)$, we obtain the following. If $H_{i+\frac{1}{2},j}=1$, then

$$d^h_{i+\frac{1}{2},j}=\frac{h^3}{24}\frac{\partial^2 p}{\partial x^2}\left(x_{i+\frac{1}{2}},y_j\right)-\frac{h^3}{24}\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial p}{\partial x}-u\right)\right]\left(x_{i+\frac{1}{2}},y_j\right)+\cdots.$$

If $0<H_{i+\frac{1}{2},j}<1$, then

$$d^h_{i+\frac{1}{2},j}=h^2\frac{1-H_{i+\frac{1}{2},j}}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}-u\right)\right]\left(x_{i+\frac{1}{2}},y_j\right)+\cdots.$$

These two relations show the lemma. □

Now we are ready to state our main result.

THEOREM 3.12 (Convergence of gradient). *Given a smooth vector field U^* , let U be its analytic Hodge projection and U^h be the numerical approximation from the Gibou-Min method, then $\left\|U-U^h\right\|_{L^2}=O\left(h^{1.5}\right)$.*

Proof. We decompose the vector field $U-U^h$ as $U-U^h=(U^*-\nabla p)-\left(U^*-Gp^h\right)$. Since G is the standard central finite difference operator, $\nabla p-Gp=O\left(h^2\right)$. Hence it is enough to show $\left\|Gp-Gp^h\right\|_{L^2}=O\left(h^{1.5}\right)$.

From Lemmas 3.5 and 3.9, we have

$$\left\langle \frac{1}{h}d^h-G\left(p-p^h\right),G\left(p-p^h\right)\right\rangle=\frac{1}{h^2}\left\langle \frac{1}{h}Dd^h-L^h\left(p-p^h\right),p-p^h\right\rangle=0.$$

Using this orthogonality, we have

$$\begin{aligned}\left\|\frac{1}{h}d^h\right\|_{L^2}^2&=\left\|\frac{1}{h}d^h-G\left(p-p^h\right)\right\|_{L^2}^2+\left\|G\left(p-p^h\right)\right\|_{L^2}^2\\&\geq\left\|G\left(p-p^h\right)\right\|_{L^2}^2.\end{aligned}$$

Then it is enough to show that $\left\|\frac{1}{h}d^h\right\|_{L^2}^2=O\left(h^3\right)$. Now we use the size estimate for d^h,

$$\begin{aligned}\left\langle \frac{1}{h}d_h,\frac{1}{h}d_h\right\rangle&=\sum_{ij}H_{i+\frac{1}{2},j}\left(d_{i+\frac{1}{2},j}\right)^2+\sum_{ij}H_{i,j+\frac{1}{2}}\left(d_{i,j+\frac{1}{2}}\right)^2\\&=\sum_{H_{i+\frac{1}{2},j},H_{i,j+\frac{1}{2}}=1}O\left(h^6\right)+\sum_{0<H_{i+\frac{1}{2},j},H_{i,j+\frac{1}{2}}<1}O\left(h^4\right)\\&=O\left(h^{-2}\right)O\left(h^6\right)+O\left(h^{-1}\right)O\left(h^4\right)=O\left(h^3\right).\end{aligned}$$

Here we used the fact that the number of inside edges, where $H=1$, grows quadratically, and the number of edges near the boundary, where $0<H<1$, linearly. □

4. Numerical test

In this section, we numerically test the Gibou-Min method in two and three dimensions to validate our analysis. The tests are performed with a uniform grid spacing $3/N$, where N is the number of grid points on one direction. Our analysis showed that the L^2-norm of the error $U-U^h$ decreases at least as fast as $h^{1.5}$. Our numerical results also show the same order, which suggests that our analysis is optimal. The associated matrix with the Gibou-Min method is sparse and symmetric positive-definite, and was solved by the Conjugate Gradient method with stopping criteria $\|r^n\|\leq\|r^0\|\cdot 10^{-12}$.

4.1. Two dimensional example

In $\Omega=\left\{(x,y)|x^2+y^2<1\right\}$, we take a vector field $U=(u,v)$ with $u(x,y)=-2xy+\frac{xy}{\sqrt{x^2+y^2}}$ and $v(x,y)=3x^2+y^2-\frac{2x^2+y^2}{\sqrt{x^2+y^2}}$, and choose a scalar variable $p(x,y)=e^{x-y}$. U was chosen such that $U\cdot\vec{n}=0$ on $\partial\Omega$ and $\nabla\cdot U=0$ in Ω. On the vector field $U^*=U+\nabla p$, the Gibou-Min method computes its discrete Hodge projection U^h. The convergence behaviors are reported in Table 4.1.

4.2. Three dimensional example

In $\Omega=\left\{(x,y)|x^2+y^2+z^2<1\right\}$, we take a vector field $U=(x^2z+3y^2z{-}2xyz,{-}x^3-xy^2)$ and a scalar variable $p(x,y,z)=e^{x-y+z}$. Note that $U\cdot\vec{n}=0$ on $\partial\Omega$ and $\nabla\cdot U=0$ in Ω. The run of the Gibou-Min method on $U^*=U+\nabla p$ is reported in Table 4.2.

grid	$\|U-U^h\|_{L^2}$	order
40^2	6.67×10^{-3}	
80^2	2.48×10^{-3}	1.42
160^2	8.14×10^{-4}	1.60
320^2	3.05×10^{-4}	1.41
640^2	1.01×10^{-4}	1.58

TABLE 4.1. *Convergence order in the two dimensional example*

grid	$\|U-U^h\|_{L^2}$	order
20^3	1.11×10^{-2}	
40^3	3.89×10^{-3}	1.51
80^3	1.31×10^{-3}	1.57
160^3	4.43×10^{-4}	1.56

TABLE 4.2. *Convergence order in the three dimensional example*

5. Conclusion

In this work, we performed convergence analyses for the Gibou-Min method that calculates the Hodge projection. We introduced a discrete integration-by-parts, and an L^2-orthogonality theorem. Using the L^2-orthogonality between the error vector $Gp-Gp^h$ and the consistency d^h, we proved the estimate $\|U-U^h\|_{L^2}=O(h^{1.5})$. According to our numerical tests, the estimate $\|U-U^h\|_{L^2}=O(h^{1.5})$ is tight.

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