# Inverse Conductivity Problem

[ A motivation to study the Cauchy problem ]

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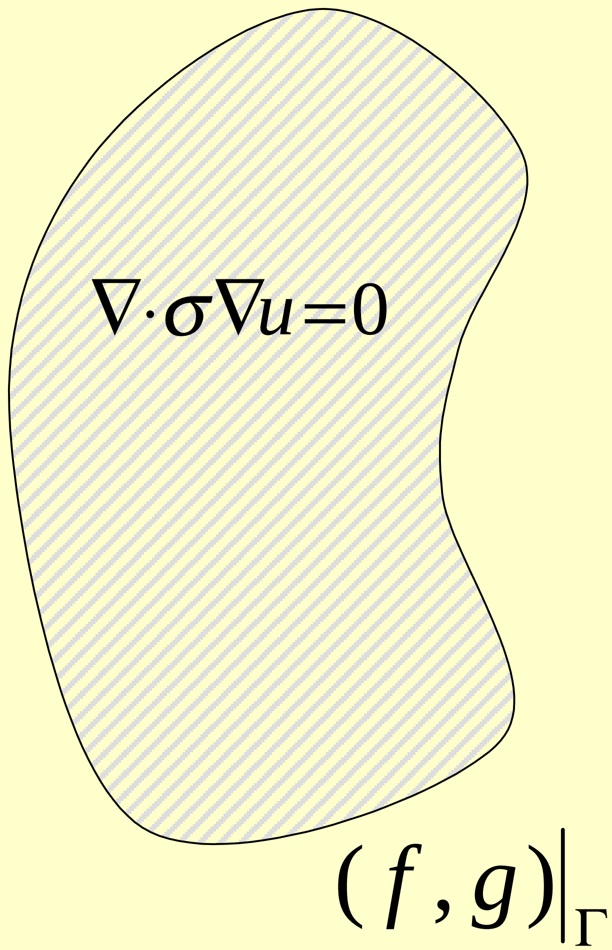
# The Cauchy Problem

(Forward) Conductivity Problem

$$For\ given\ \sigma(x),\ x \in \Omega,$$

$$find\ u(x)\ in\ \Omega\ s.t.$$

$$u = f\ OR\ \frac{\partial u}{\partial n} = g\ on\ \partial\Omega$$




The Cauchy Problem

$$For\ given\ \sigma(x) = \sigma_0\ in\ \Omega,$$

$$find\ u(x)\ in\ \Omega\ s.t.$$

$$u = f\ AND\ \frac{\partial u}{\partial n} = g\ on\ \Gamma \subset \partial\Omega.$$

# Physical Background

(Forward) Conductivity Problem

$$\textit{For given } \sigma(x),\ x \in \Omega,$$

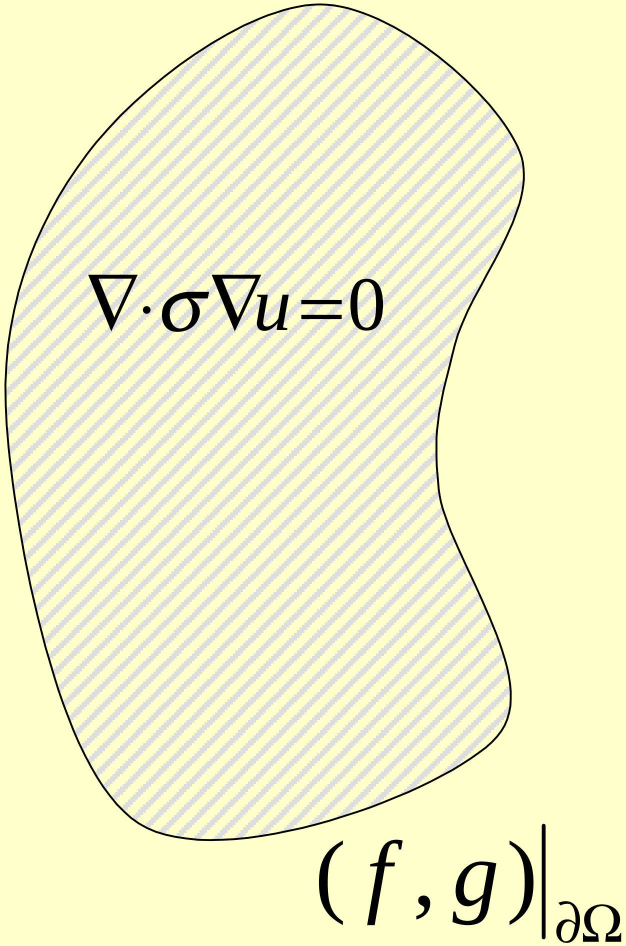
$$\textit{find } u(x) \textit{ in } \Omega \textit{ s.t.}$$

$$u = f \;\; OR \;\; \frac{\partial u}{\partial n} = g \;\; on \;\; \partial\Omega$$

Inverse Conductivity Problem

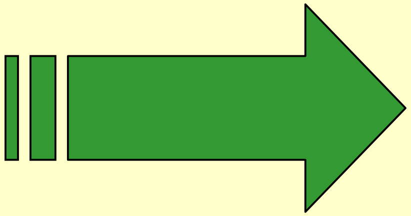
$$\textit{Find } \sigma(x) \textit{ from given}$$

$$u = f \;\; AND \;\; \frac{\partial u}{\partial n} = g \;\; on \;\; \partial\Omega$$

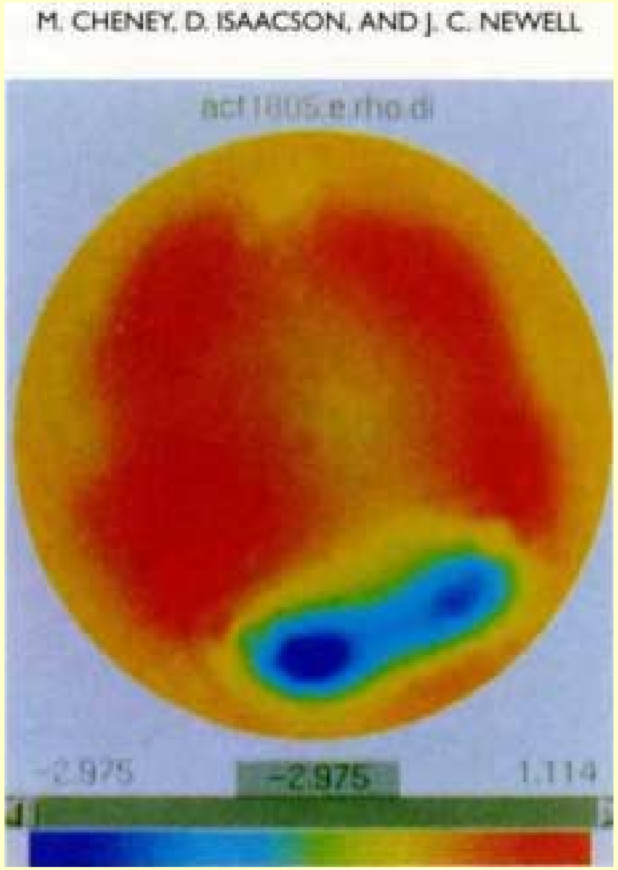


# Electrical Impedance Tomography (EIT)

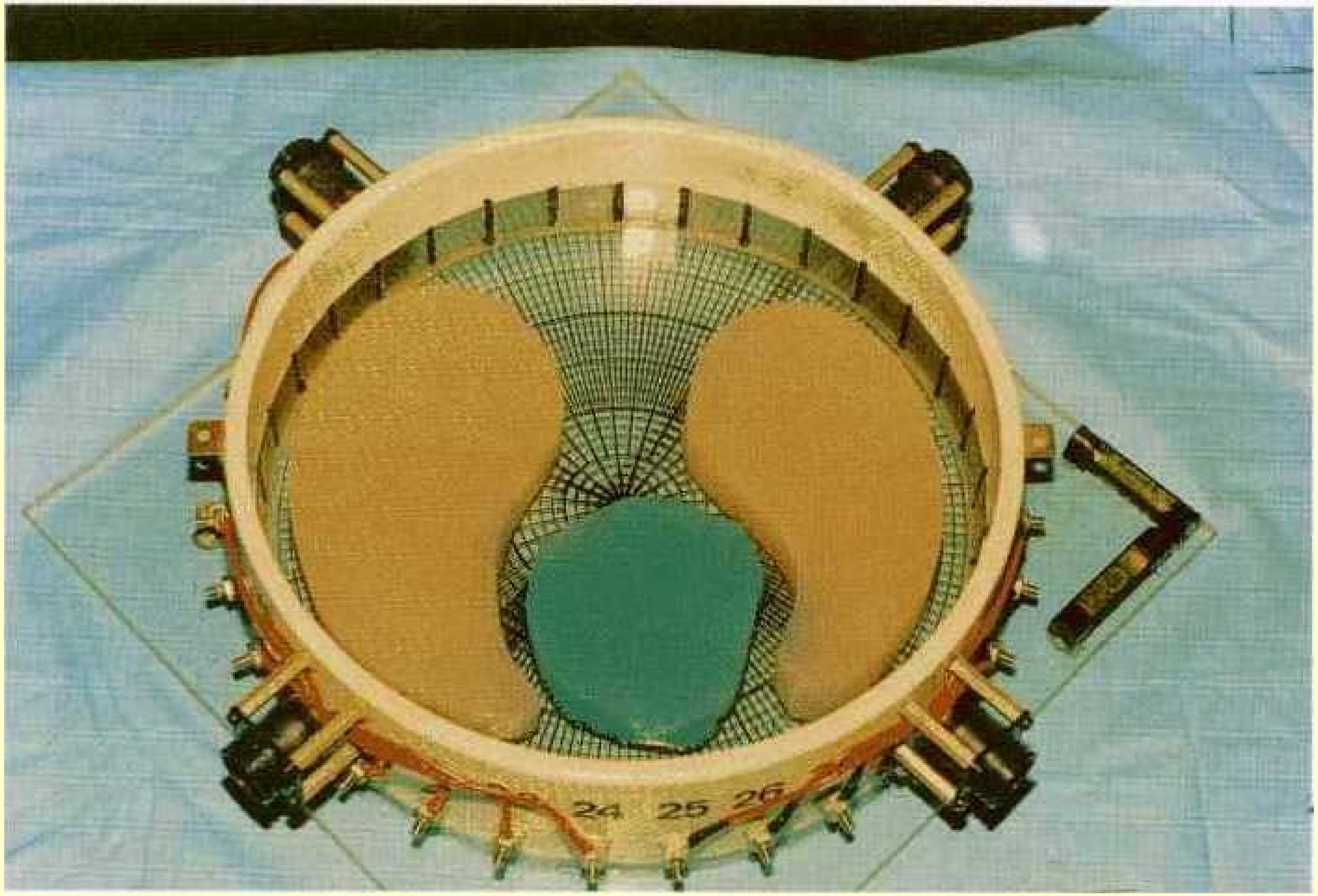
Impedance
~Voltage/Current



Tomography

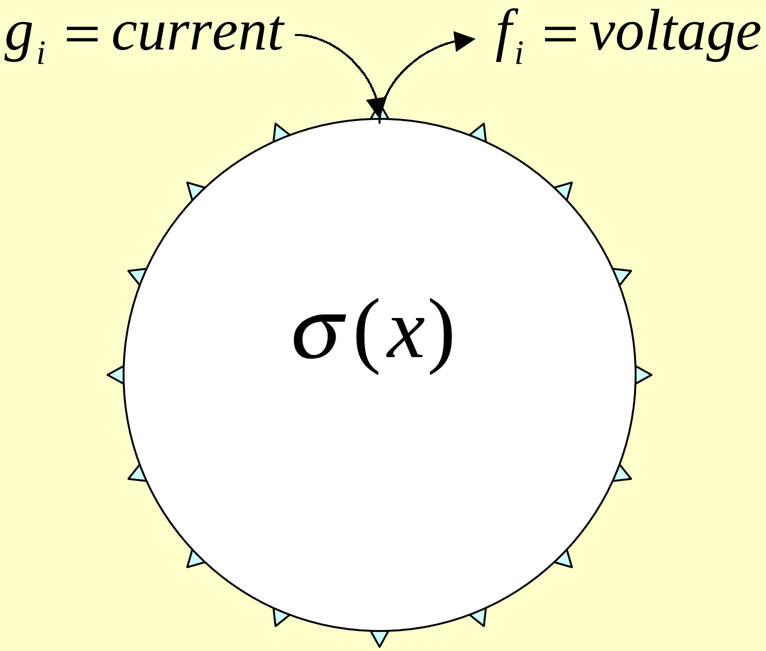


# A model problem with heart and lung



# Methods of EIT/ICP

Objective $\underset{\sigma}{Min} \sum \left| f_k - u_\sigma(x_k) \right|^2$ where $\nabla \cdot \sigma \nabla u_\sigma = 0, \ \frac{\partial u}{\partial n} = g$




- Single Measurement
  $\Rightarrow$ minimum information
- Many Measurement
  $\Rightarrow$ full information (?) on $\sigma$
- Infinite Measurement
  $\Rightarrow$ mathematical theory

# Numerical Obstacles for EIT/ICP

Objective $\underset{\sigma}{Min} \sum \left| f(x) - u_\sigma(x) \right|_{L_2(\Omega)}$ where $\nabla \cdot \sigma \nabla u_\sigma = 0, \frac{\partial u}{\partial n} = g$

Guss $\sigma(x) \rightarrow$ Solve $\nabla \cdot \sigma \nabla u_\sigma = 0, \frac{\partial u_\sigma}{\partial n} = g \rightarrow$ Find better $\tilde{\sigma}$

- Strongly nonlinear: $\sigma \in L_2(\Omega) \rightarrow u_\sigma \in L_2(\partial\Omega)$

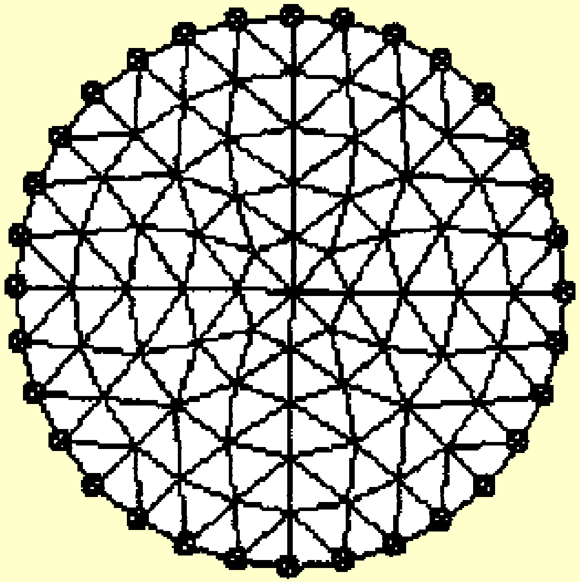
Solve $\nabla \cdot \sigma \nabla u_\sigma = 0, \frac{\partial u_\sigma}{\partial n} = g \rightarrow$ Find better $\tilde{\sigma}$ using $u_\sigma - f$

- Ill-posed problem: $\sigma \neq \tilde{\sigma} \rightarrow u_\sigma \approx u_{\tilde{\sigma}}$

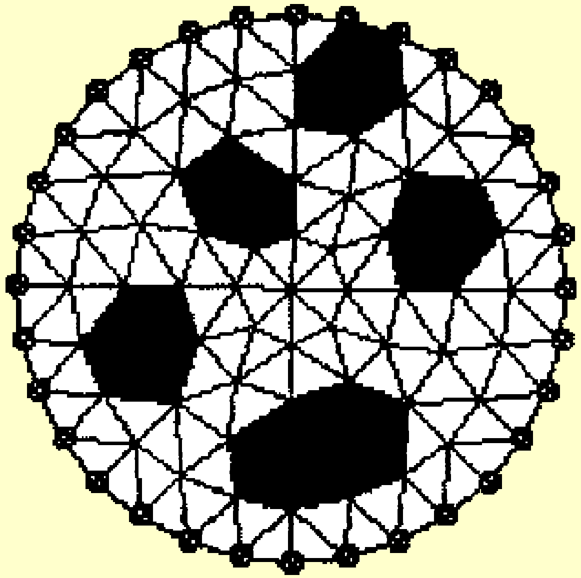
Stop condition $u_\sigma - f \cong 0 \rightarrow$ Regularity restrictiuon on $\sigma$

# A case study

Finite Element Mesh



High Conductivity Inc.



# Finite Element Solution for σ

- Minimize $E(\sigma) = \sum_k \left| f_k - u_\sigma(x_k) \right|^2$

$$0 = \frac{\partial E(\sigma)}{\partial \sigma_i} = 2\sum_k (u_{\sigma_i}(x_k) - f(x_k)) \cdot \frac{\partial u_\sigma}{\partial \sigma_i}$$

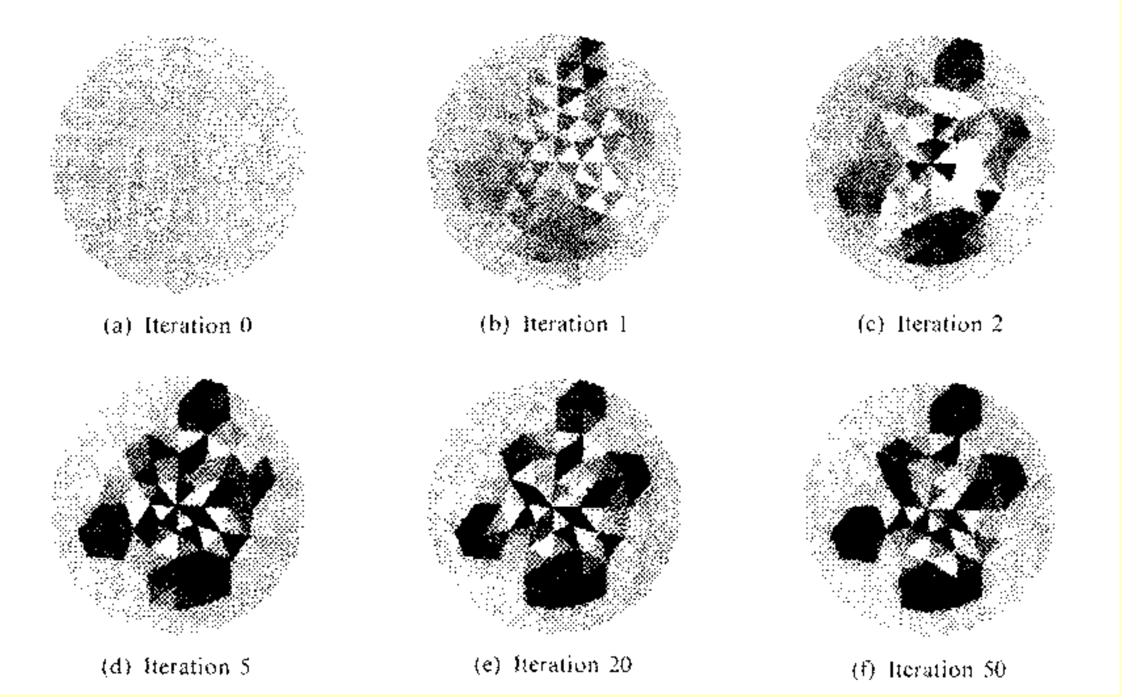
- Newton's method to solve the non-linear eq.

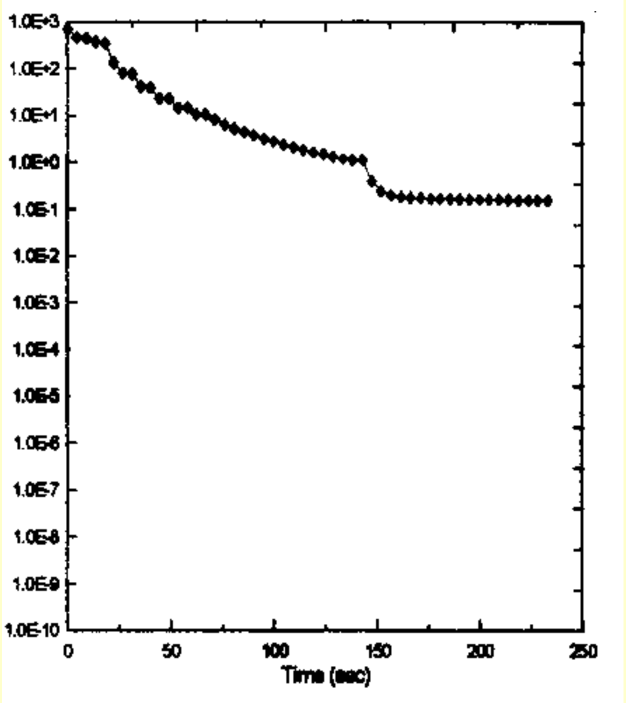
$$\delta\sigma \approx -H^{-1} J^T (u_{\sigma_i}(x_k) - f(x_k))$$

$$\text{where} \quad H \approx \left[\frac{\partial u_\sigma}{\partial \sigma_i}\right]^T \left[\frac{\partial u_\sigma}{\partial \sigma_i}\right], \; J \approx \frac{\partial u_\sigma}{\partial \sigma_i}$$

# Ill-posed Nonlinear

- 50 iterations



# Fundamental Difficulty of ICP

- Ill-poseness of Cauchy Problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \Gamma_D, \qquad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_N,$$

$$u(x) = \int_{\partial\Omega} \Phi(x,y)\sigma(y)\, ds_y,$$

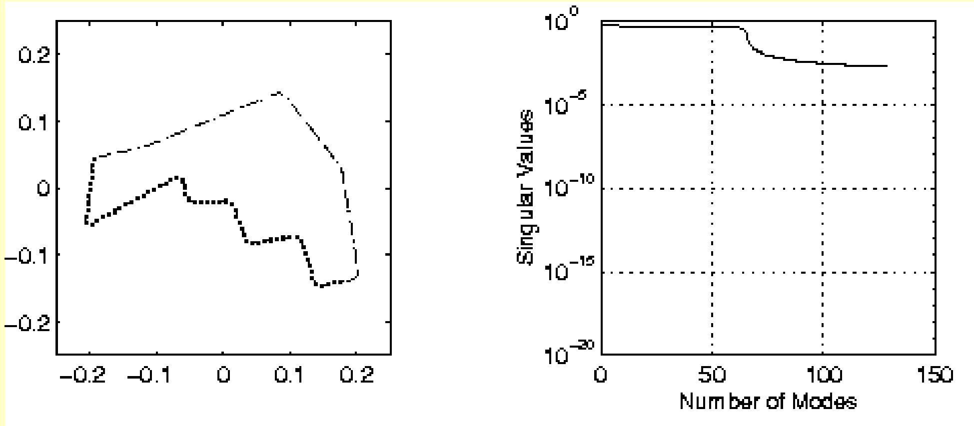
$$\mathcal{C}\sigma := \begin{pmatrix} \mathcal{S}_\Omega \sigma|_{\Gamma_D} \\ \left(-\frac{1}{2} I + \mathcal{K}_\Omega^*\right) \sigma|_{\Gamma_N} \end{pmatrix}.$$

Cauchy Problem ⟷ $\mathcal{C}[\sigma] = \begin{pmatrix} f \\ g \end{pmatrix}$

# Well-posed vs. Ill-posed

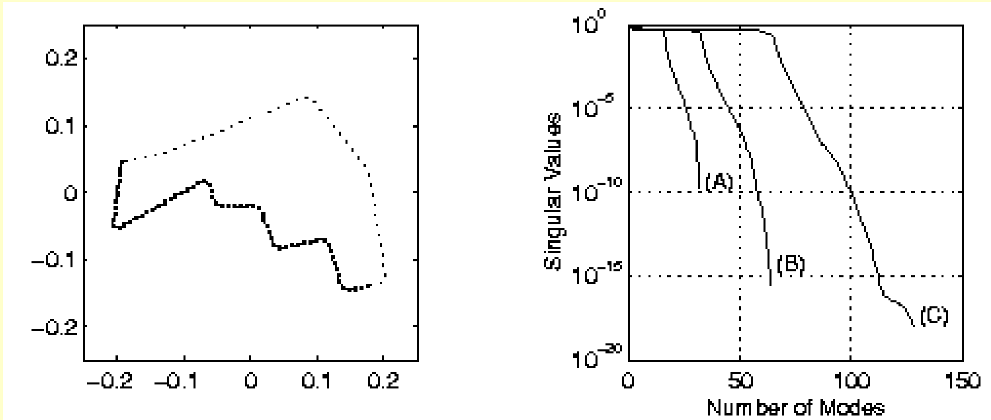
Well-posed

Mixed boundary value problem




Ill-posed

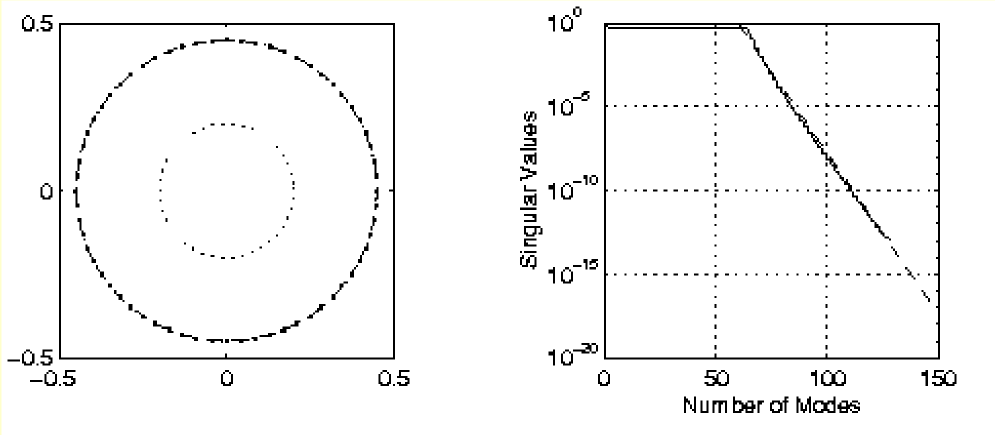
Cauchy (harmonic extention) problem



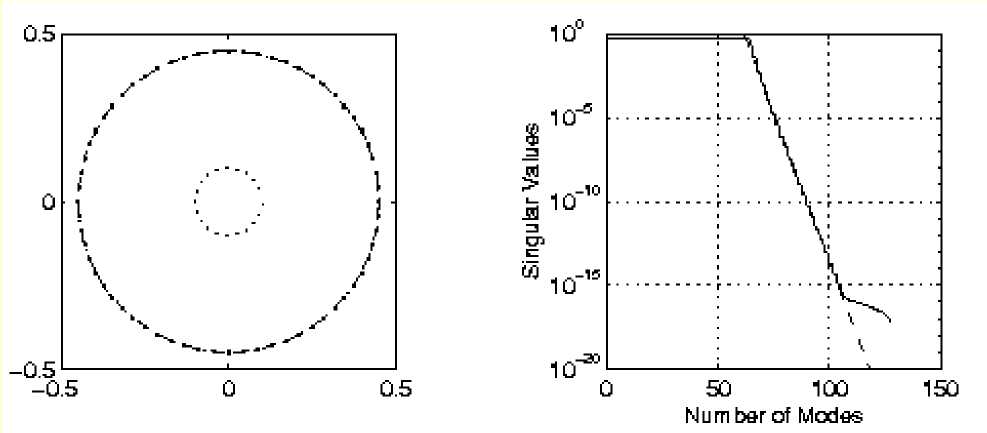
# Singular Values of Cauchy Operator

$$\mu_R^m = O(1) \quad and \quad \mu_r^m = O\left(\left(\frac{r}{R}\right)^m\right)$$

$$\Omega = B_{0.45} \setminus B_{0.2}$$

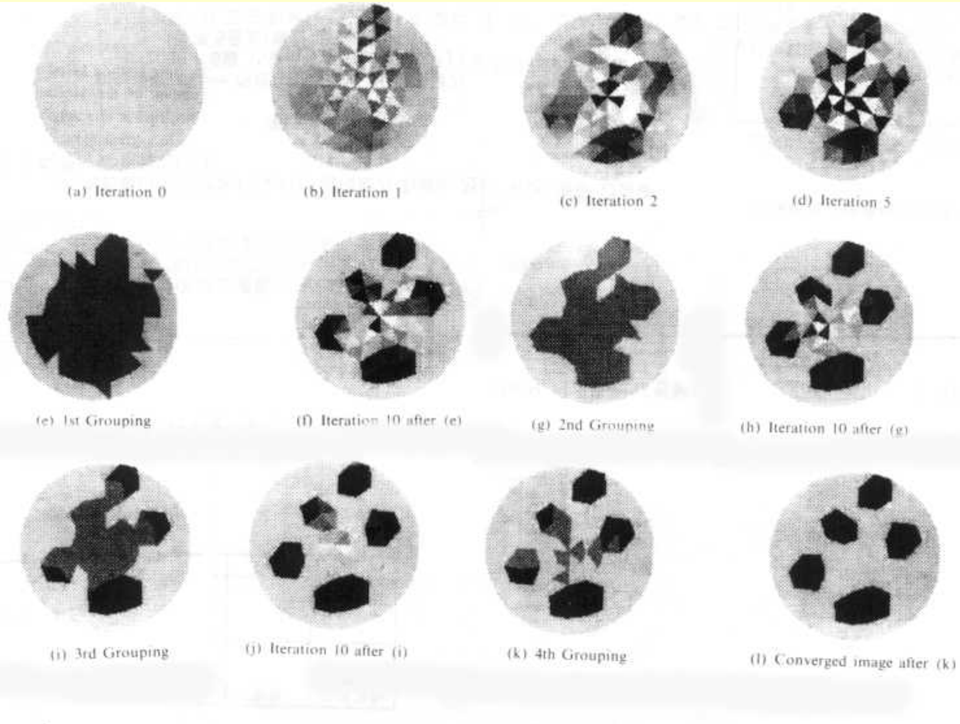


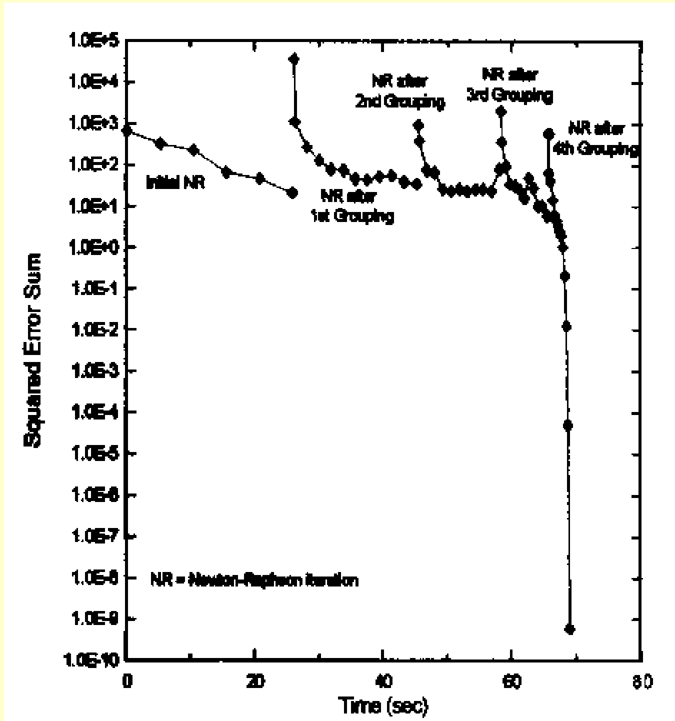

$$\Omega = B_{0.45} \setminus B_{0.1}$$



# Restrictions on conductivity Profile

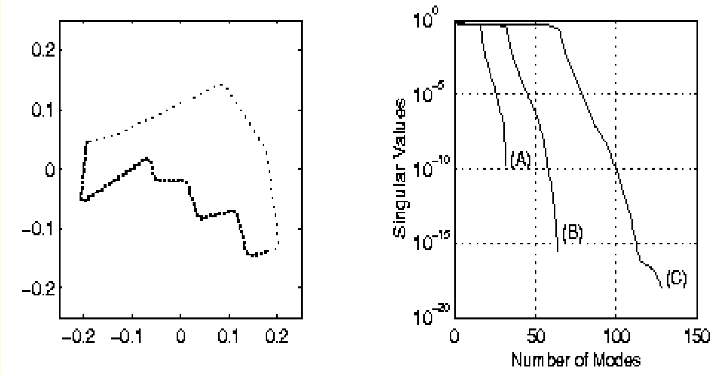
- Mesh Grouping Algorithm

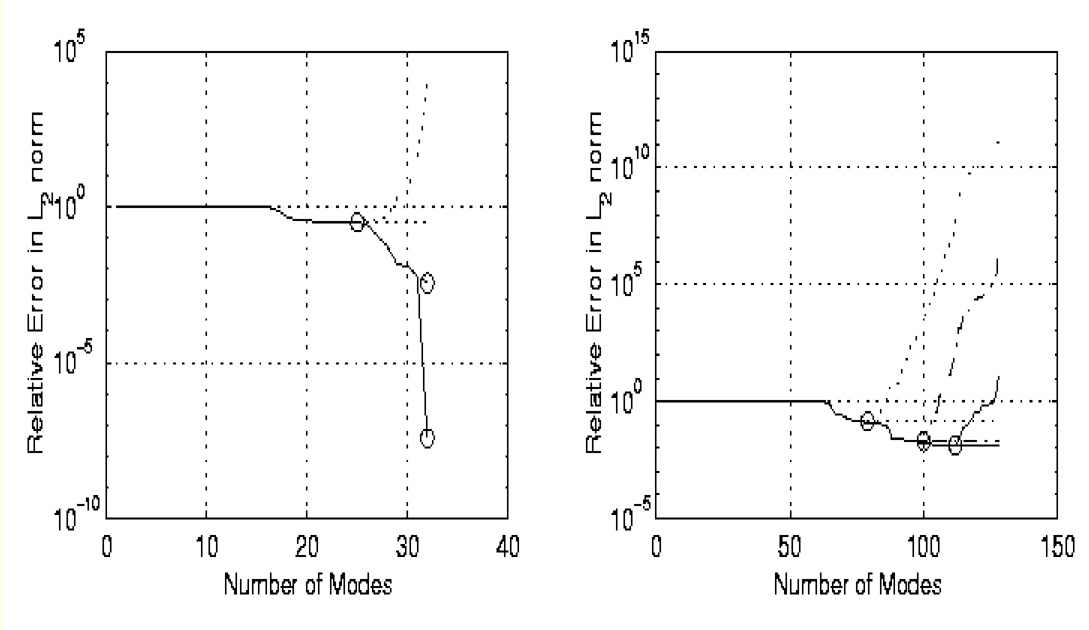


# Requirements for Cauchy solver

- Numerically Stable
  (small condition number)
- Numerically Accurate
  (small computational error)




(1) Dotted : Error= $10^{-5}$

(2) Dash-Dotted : Error= $10^{-10}$

(3) Solid : Error=$10^{-15}$

(A) Small condition number (B) Large condition number