Chapter 3. Image Transforms

- **Fourier Transform of 1 and 2 continuous**
- **Discrete Fourier Transform**
- **Properties of 2-D Fourier Transform**
- **FFT**
- **Other Transforms**

Transform theory: key role in image processing
→ 2-D transforms to image enhancement, restoration, encoding, …

3.1 Introduction to the Fourier Transform

- Notation of FT pair:

$$\mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2xu}\, dx$$

- inverse FT

$$\mathcal{F}^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\, e^{-j2\pi ux}\, du$$

if f(x) is continuous and integrable and F(u) is integrable

- $f(x) : real \xrightarrow{FT} F(u) : complex$

$$\begin{aligned} F(u) &= R(u) + jI(u) \\ &= |F(u)| e^{j\phi(u)} \end{aligned}$$

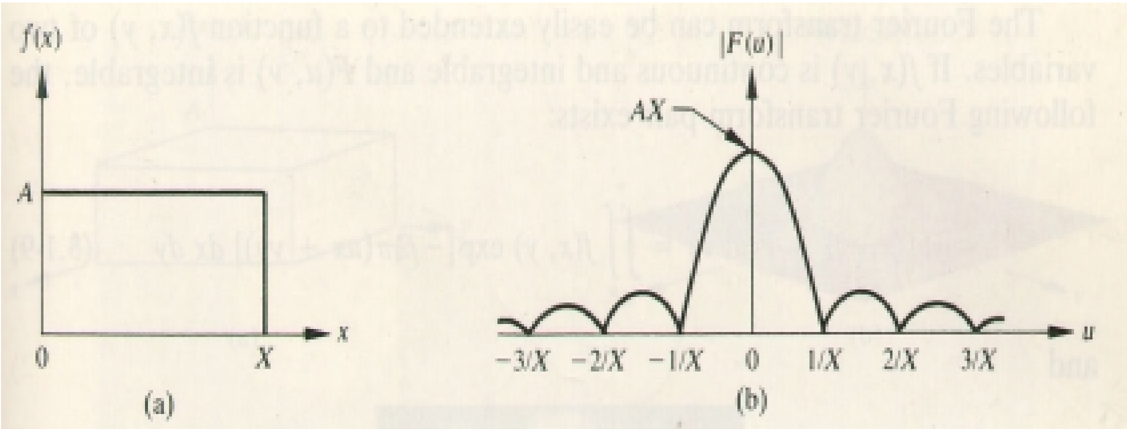
$$where \quad |F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2} : Fourior\ spectrum\,(magnitude)$$

$$\phi(u) = \tan^{-1}\frac{I(u)}{R(u)} \qquad : phase$$

- square of spectrum :

$$\begin{aligned} P(u) &= |F(u)|^2 \\ &= R^2(u) + I^2(u) \quad : power\ spectrum\ of\ f(x) \quad (spctral\ density) \end{aligned}$$

- example:



- 2-D FT pair:

  - extension of 1-D FT;

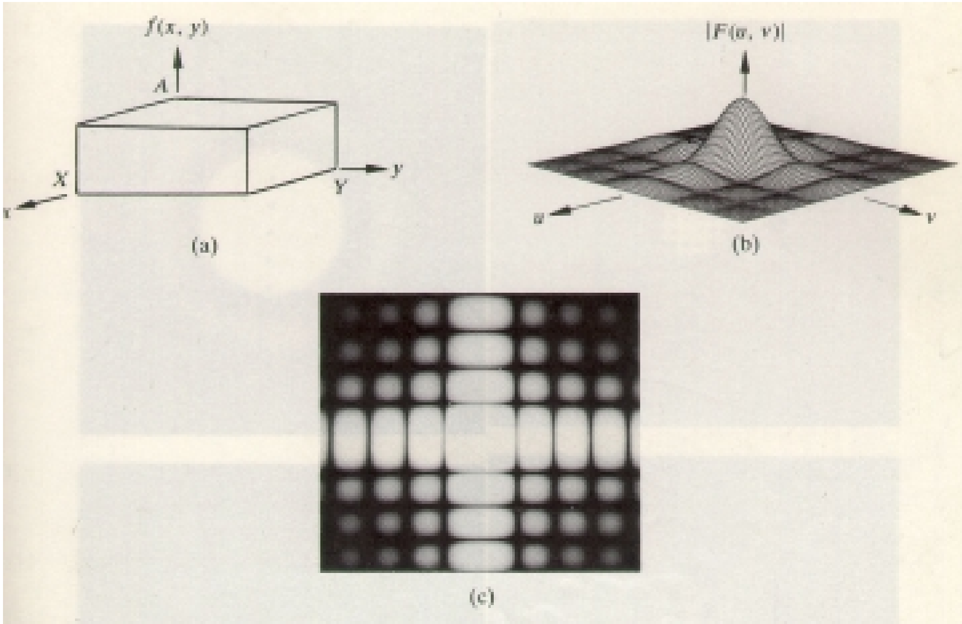
$$\mathcal{F}\{f(x,y)\} = F(u,v) = \int\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dxdy$$

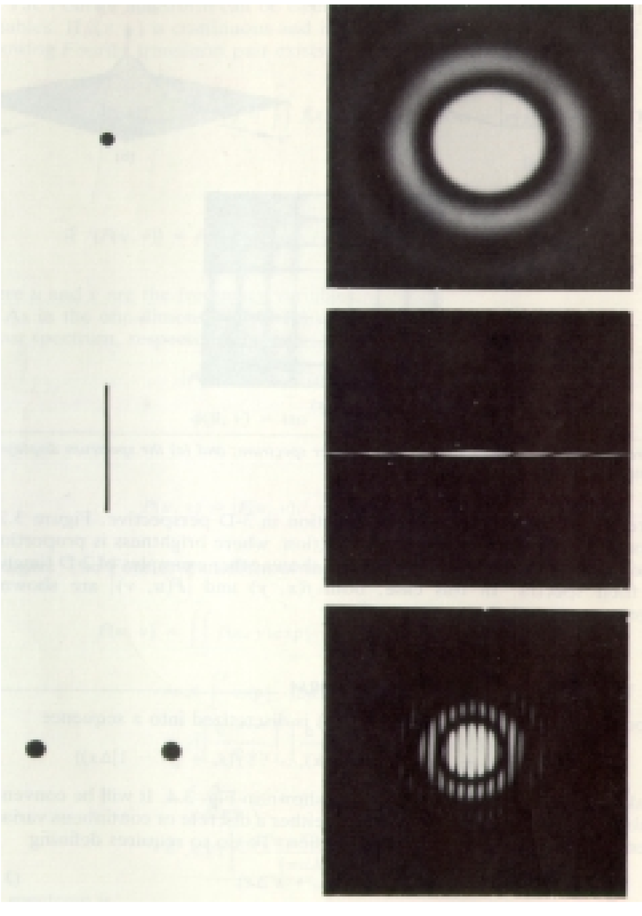
$$\mathbf{F}^{-1}\{F(u,v)\} = f(x,y) = \int\int_{-\infty}^{\infty} F(u,v)\, e^{+j2\pi(ux+vy)}\, dudv$$

- spectrum, phase and power spectrum:

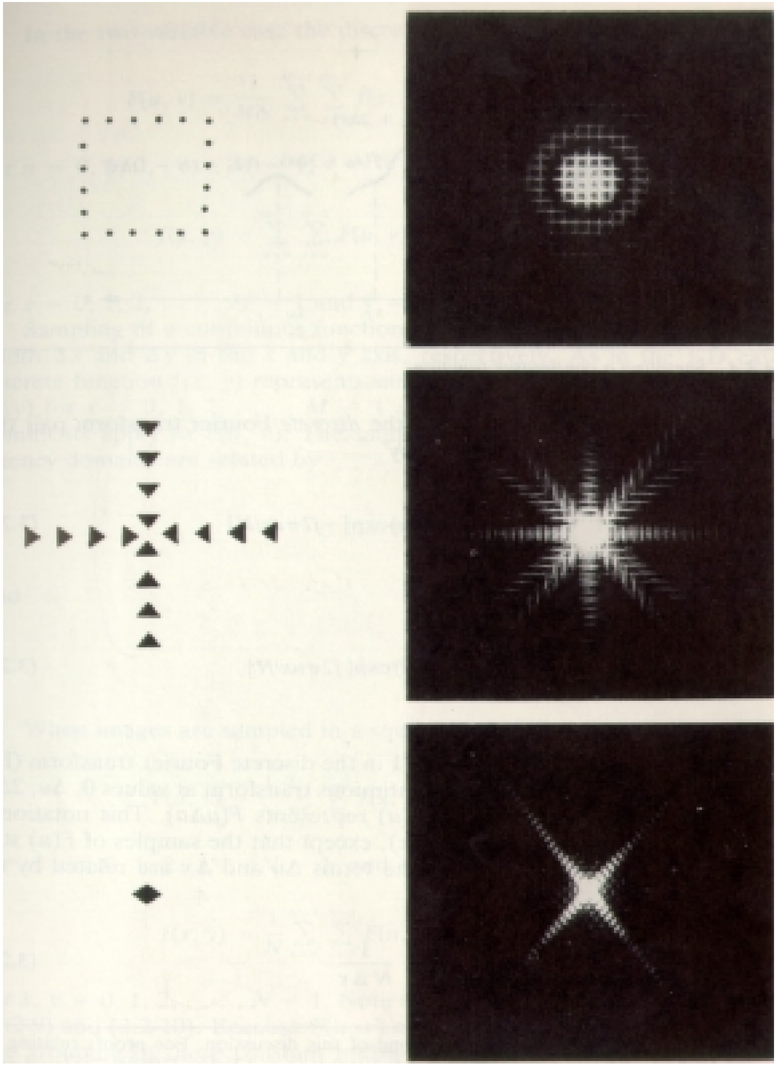
  - $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$; Fourier Spectrum
  - $\phi(u,v) = \tan^{-1}\dfrac{I(u,v)}{R(u,v)}$; Phase Spectrum
  - $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$; Power Spectrum

Example:





(a)



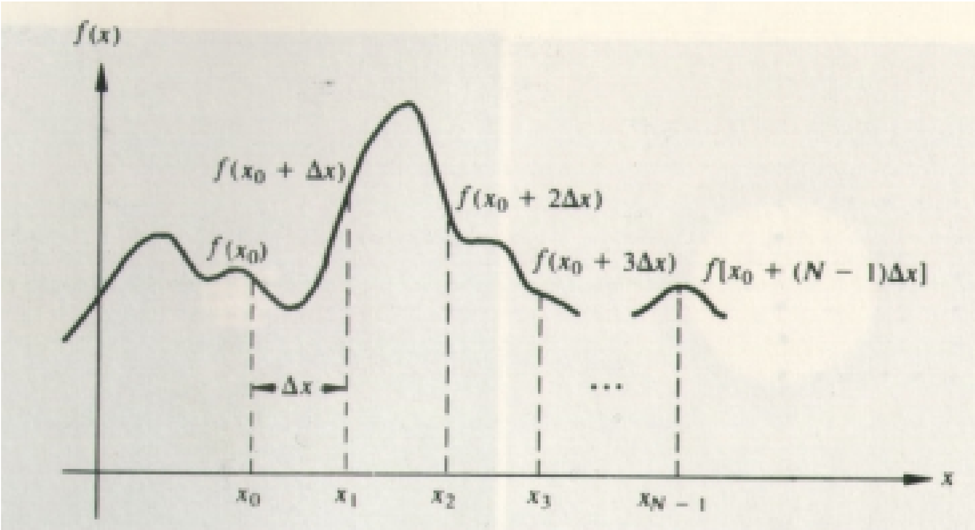
3.2 The Discrete Fourier Transform (DFT)

- Sampling:

$$f(x) \rightarrow \{f(x_0), f(x_0+\Delta x), \cdots, f(x_0+(N+1)\Delta x)$$

uniformly spaced samples

$f(0), f(1), \cdots f(N-1)$



- DFT pair:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp(-j2\pi\, ux/N)$$

$$for \quad u = 0,1,\cdots,N-1$$

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp(j2\pi\, ux/N)$$

$$for \quad x = 0,1,\cdots,N-1$$

- values u=0,1,…,N-1

  : correspond to samples of continuous transform at values $0,\Delta u, 2\Delta u,\cdots,(N-1)\Delta u$

$$F(u) \rightarrow F(u\Delta u)$$

$$\Delta u = \frac{1}{N\Delta x}$$

- 2-D DFT:

- $$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp(-j2\pi\,(ux/M + vy/N)$$

$$for \quad u = 0,1,\cdots,N-1 \quad v = 0,1,\cdots,N-1$$

- $$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp(j2\pi(ux/M + vy/N)$$

  $$for \quad u = 0,1,\cdots,N-1 \quad v = 0,1,\cdots,N-1$$

- sampling : 2-D grid

  $$\Delta u = \frac{1}{M\Delta x}, \qquad \Delta v = \frac{1}{N\Delta y}$$

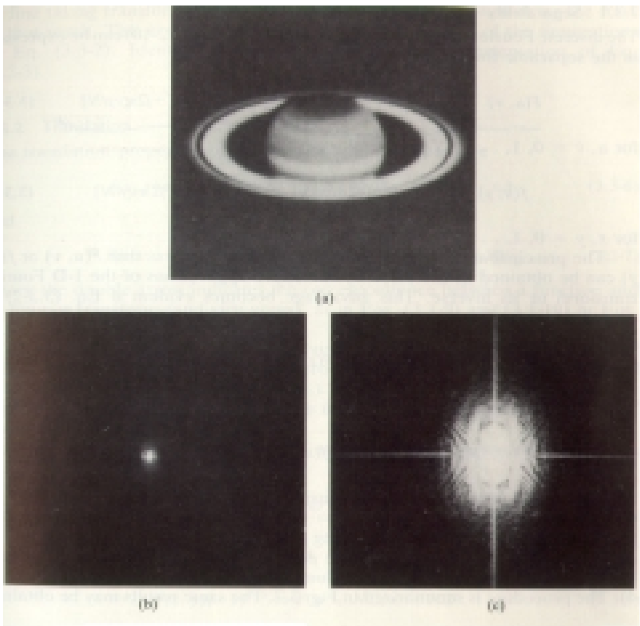
- for square array, M=N ;

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \exp(-j2\pi(ux+vy)/N)$$

$$f(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) \exp(j2\pi(ux+vy)/N)$$

3.3 Some Properties of the 2-D FT

- dynamic range of Fourier spectra :
  - much higher than the typical display device
  - range : $[0, 2.5\times10^{6}]$



- only the brightest parts are visible on the display screen

- log compression ;

$$D(u,v) = c\log[1+|F(u,v)|]$$

✓ Scaling constant

3.3.1 Separability

- $$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\exp(-j2\pi\, ux/N)\sum_{y=0}^{N-1} f(x,y)\exp(-j2\pi\, vy/N)$$

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\exp(+j2\pi\, ux/N)\sum_{n=0}^{N-1} F(u,v)\exp(+j2\pi\, vy/N)$$

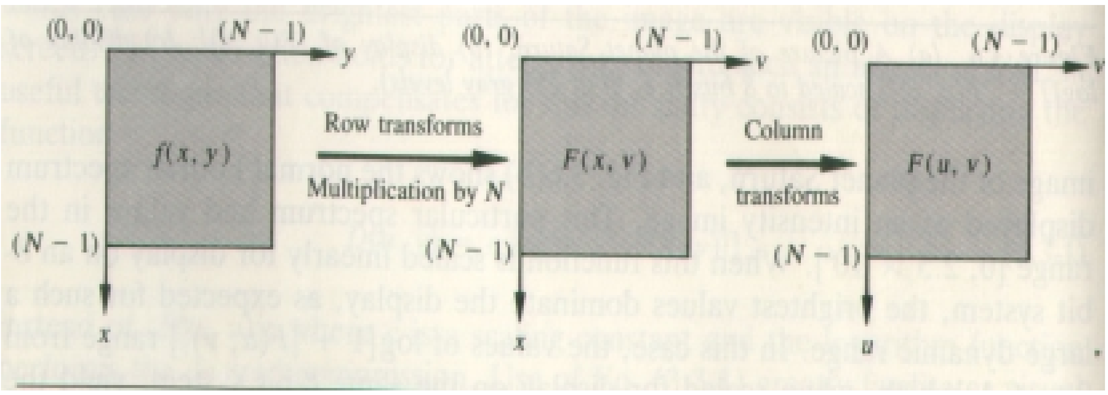
: obtained in two steps by successive applications of 1-D FT

or

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x,v)\exp(-j2\pi\, ux/N)$$

where $F(x,v) = N[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp(-j2\pi\, ux/N)]$

: 1-D transform



- 2-D FT :

    i. 1-D transform along each row of f(x,y)
    ii. multiply by N
    iii. 1-D transform along each column of F(x,v)

3.3.2 Translation

- $f(x,y)\exp(j2\pi(u_0x+v_0y/N) \leftrightarrow F(u-u_0,v-v_0)$

  $f(x-x_0,y-y_0) \leftrightarrow F(u,v)\exp(-j2\pi(u_0x+v_0y/N)$

- $u_0=v_0={N \over 2}$;

  $\exp(j2\pi(u_0x+v_0y)/N)=e^{j\pi(x+y)}=(-1)^{x+y}$

  $\rightarrow f(x,y)(-1)^{x+y} \leftrightarrow F(u-\frac{N}{2},v-\frac{N}{2})$

3.3.3 Periodicity and Conjugate Symmetry

- periodicity : DFT and its inverse are periodic with period N :

  $$F(u,v)=F(u+N,v)=F(u,v+N)=F(u+N,v+N)$$

- conjugate symmetry :
    - if f(x,y) is real,

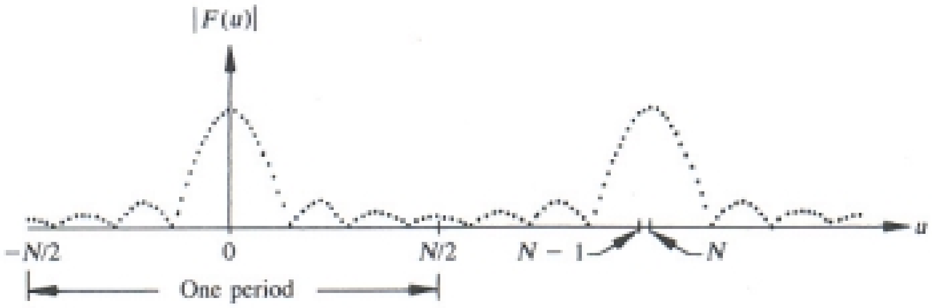
      $F(u,v)=F^*(-u,-v)$

      or $|F(u,v)|=|F(-u,-v)|$: centered on the origin

- display of one full period :

  : move the origin of the transform to the point $u={N \over 2}$

  $f(x)(-1)^x$



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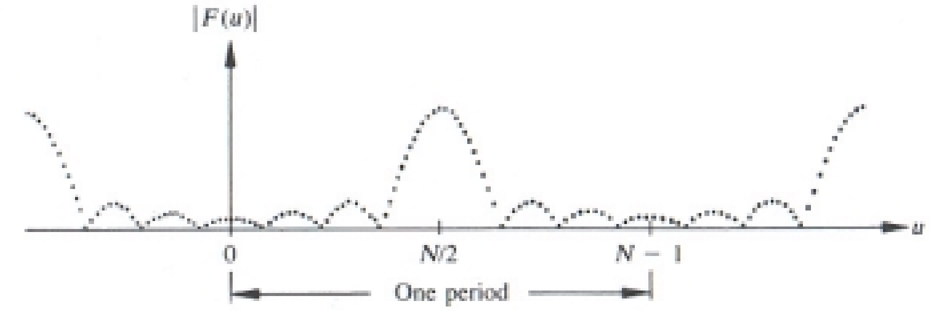

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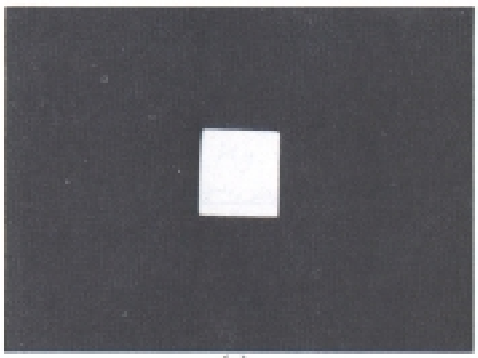

(a)

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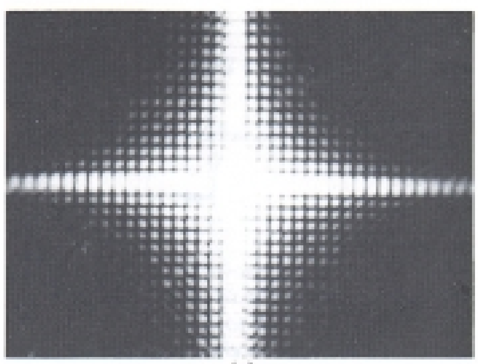


(c)

3.3.4 Rotation

- in polar coordinate :

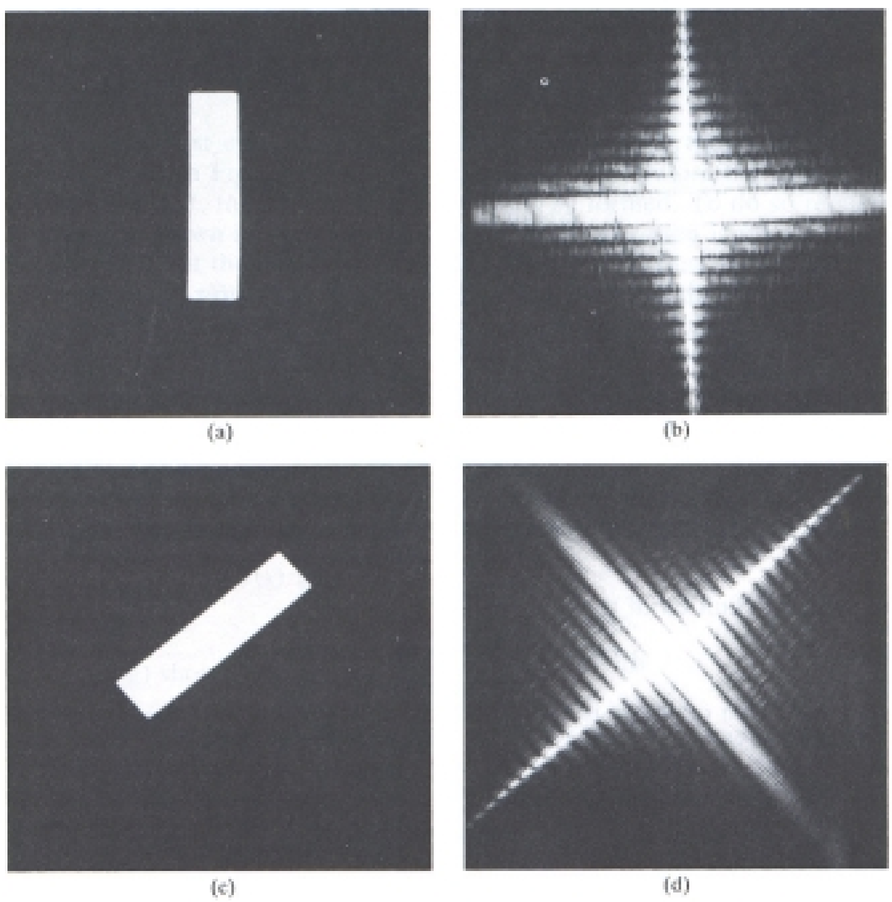
$$x = r\cos\theta, \quad y = r\sin\theta, \quad u = \omega\cos\phi, \quad v = \omega\sin\phi$$

$$(\; , \;) \rightarrow \; (r,\theta)$$

$$F(u,v) \rightarrow F(\omega,\phi)$$

- FT pair for rotation function :

  - $f(r,\theta+\theta_0) \leftrightarrow F(\omega,\phi+\theta_0)$
  - rotating f(x,y) by angle θ_0 $\rightarrow$ rotating F(u,v) by angle θ_0



3.3.5 Distributivity and scaling

- distributivity :

$$\mathrm{F} \quad y) + {}_{2} \quad y)\} = \mathrm{F} \quad y)\} + \mathrm{F}\{ {}_{2} \quad y)\}$$

$$\mathrm{F}\{f_1(x,y)\cdot f_2(x,y)\} \neq \mathrm{F}\{f_1(x,y)\}\cdot \mathrm{F}\{f_2(x,y)\}$$

- scaling :

$$) \leftrightarrow \quad v)$$

$$b \ \ ) \leftrightarrow \frac{\quad}{b} \quad - \frac{\ }{b})$$

3.3.6 Average Value

- average value of 2-D discrete function

$$\bar{\ } \quad) = \frac{\quad}{\ } \sum_{=}^{N-1} \sum_{=}^{N-1} \quad)$$

$$= \frac{\ }{\ } F \quad)$$

3.3.7 Laplacian

- Laplacian of 2-D function f(x,y) :

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\mathrm{F} \ \ \nabla^2 \quad) \ = - \ \ \pi)^2 \quad {}^2 + v^2) \quad v)$$

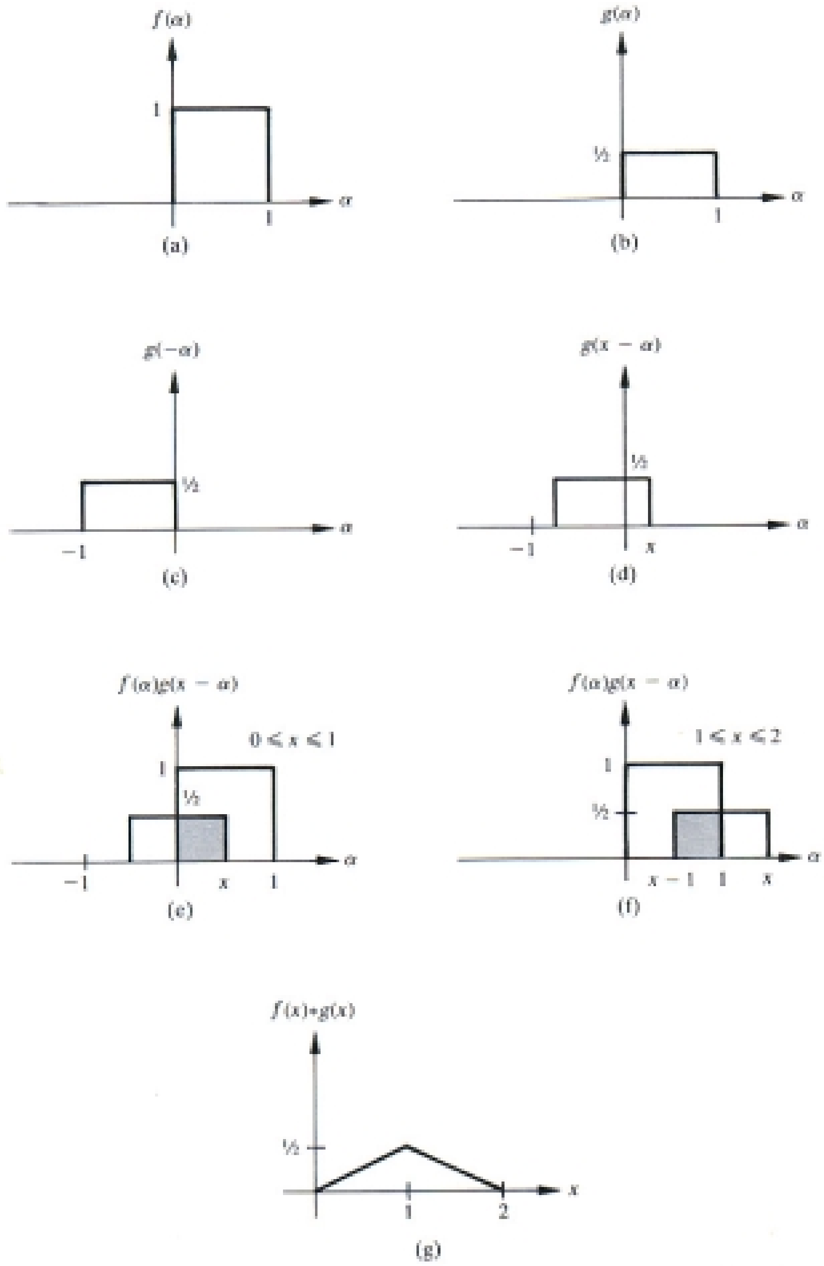
: useful for outlining edges in an image

3.3.8 Convolution and Correlation

- Convolution :

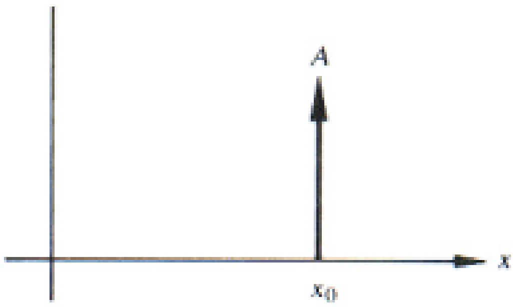
$$- \quad) * \quad) = \int_{-\infty}^{\infty} \quad \alpha) \quad -\alpha) d\alpha$$

ex.

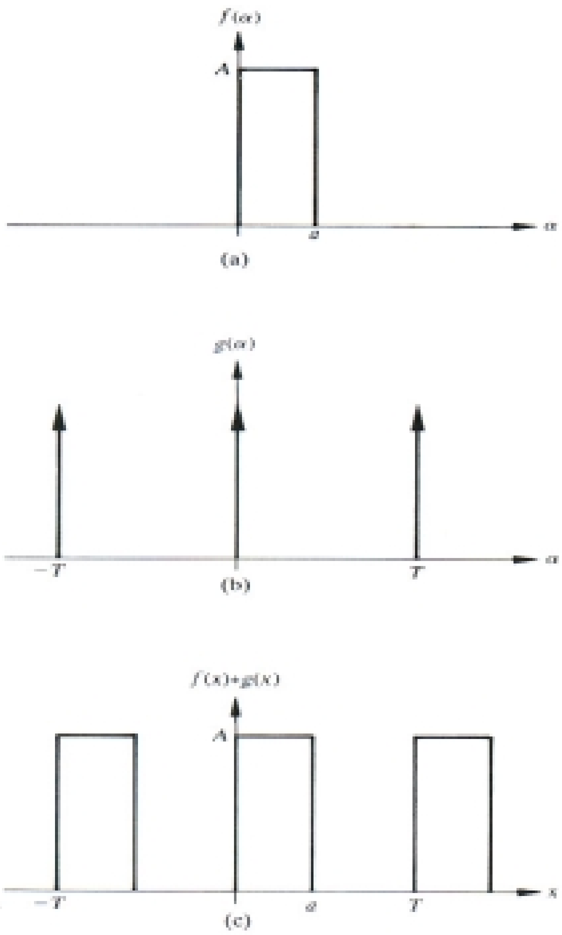


- $\int_{-\infty}^{\infty} \quad x)\delta \;\; x - x_0)x \;\; = \quad x_0)$

$$\int_{-\infty}^{\infty}\delta \;\; x - x_0)dx = \int_{x_0^-}^{x_0^+}\delta \;\; x - x_0)dx = 1$$



- ex.



- in freq. Domain, convolution theorem ;

$$f \;\; x) * \quad x) \leftrightarrow \quad u) \quad u)$$

$$f(x)g(x) \leftrightarrow F(u) * G(u)$$

- In DFT ;

f(x), g(x) : periodic some period M

$M \geq A + B - 1$

: no wraparound error

- appending zero to the samples ;

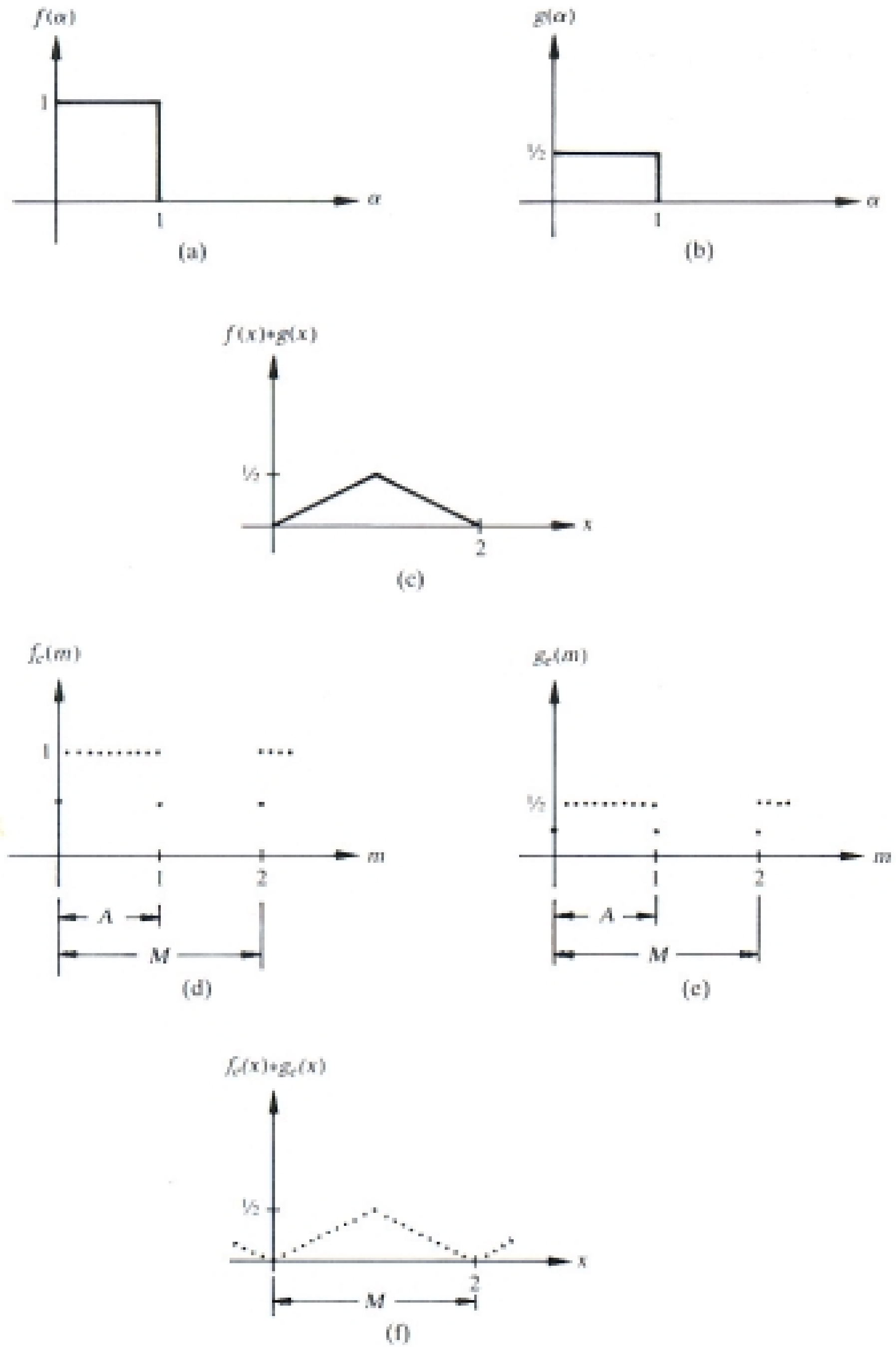
$$f_e(x) = \{ \begin{matrix} f(x) & 0 \leq x \leq A-1 \\ 0 & A \leq x \leq M-1 \end{matrix}$$

$$g_e(x) = \{ \begin{matrix} g(x) & 0 \leq x \leq B-1 \\ 0 & B \leq x \leq M-1 \end{matrix}$$

$$f_e(x) * g_e(x) = \frac{1}{M} \sum_{m=0}^{M-1} f_0(m) g_e(x-m)$$

for x=0, 1, …, M-1

ex.



- 2-D convolution ;

$$f(x,y)*g(x,y)=\iint_{-\infty}^{\infty} f(\alpha,\beta)g(x-\alpha,y-\beta)d\alpha d\beta$$

$$f(x,y)*g(x,y)\leftrightarrow F(u,v)G(u,v)$$

$$f(x,y)g(x,y) \leftrightarrow F(u,v) * G(u,v)$$

- array size ;

  ✓ $f(x,y) \quad : \quad A \times B$
  $g(x,y) \quad : \quad C \times D$

  ✓ to avoid wraparound error

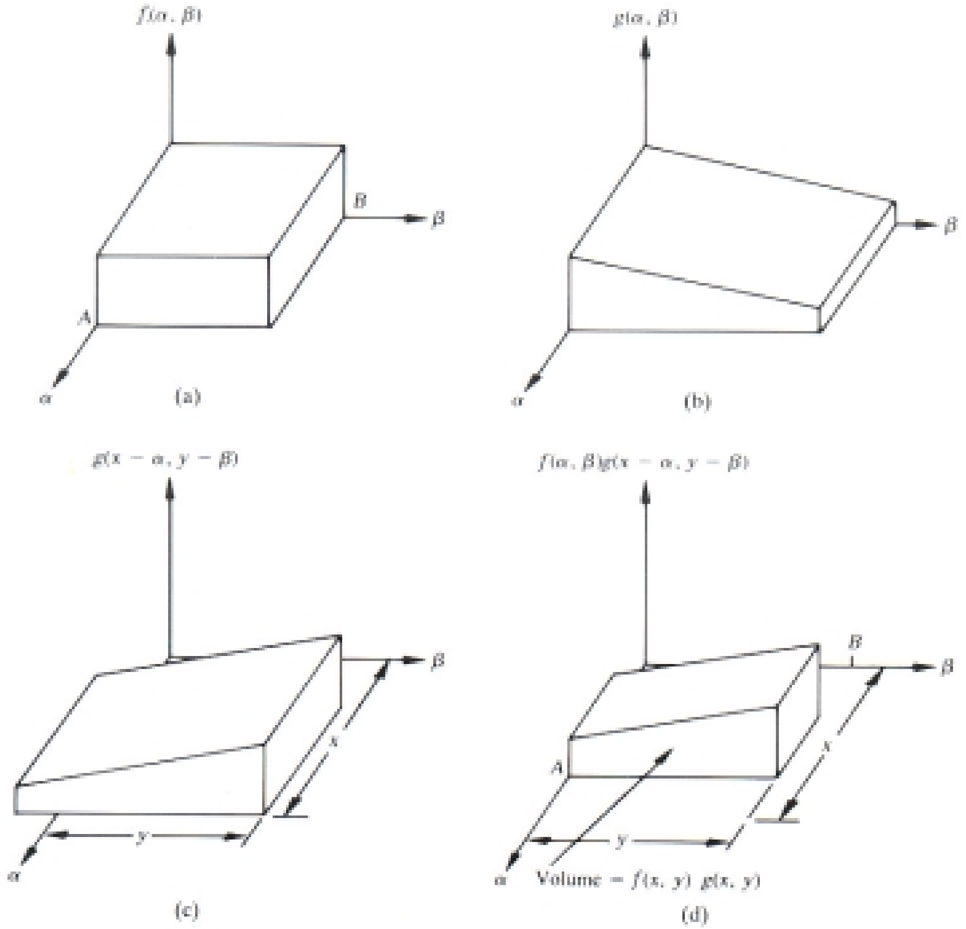
  $M \geq A + C - 1$
  $N \geq B + D - 1$

  ✓ extending f(x,y), g(x,y) ;

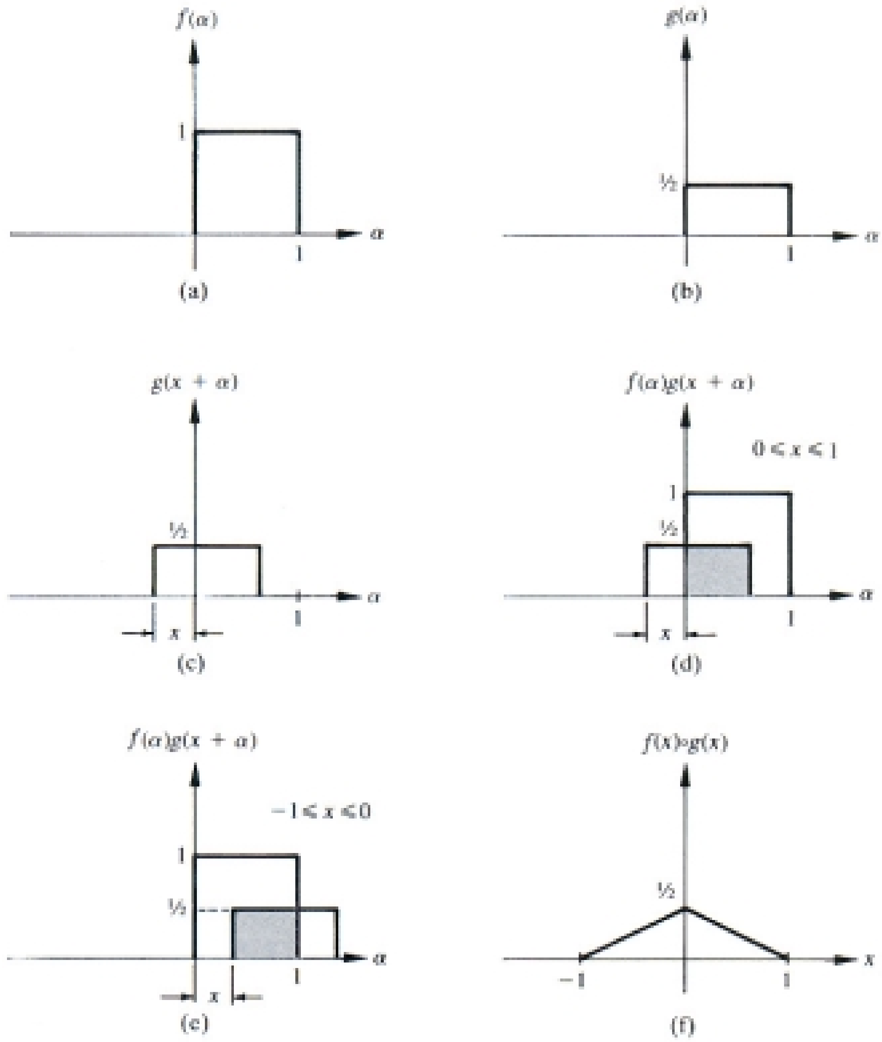
$$f_e(x,y) = \{ \begin{matrix} f(x,y) & 0 \leq x \leq A-1, \, and \; 0 \leq y \leq B-1 \\ 0 & A \leq x \leq M-1, \, and \; B \leq y \leq N-1 \end{matrix}$$

$$g_e(x,y) = \{ \begin{matrix} g(x,y) & 0 \leq x \leq C-1, \, and \; 0 \leq y \leq D-1 \\ 0 & A \leq x \leq M-1, \, and \; D \leq y \leq N-1 \end{matrix}$$

$$f_e(x,y) * g_e(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) g_e(x-m, y-n)$$



- correlation of 2 continuous functions f(x) and g(x) :
  - $f(x) \circ g(x) = \int_{-\infty}^{\infty} f^*(\alpha) g(x+\alpha) d\alpha$
  - not folded to the origin
  - if f(x)=g(x), autocorrelation
  - if not, crosscorrelation



- discrete equivalent

$$f_e(x) \circ g_e(x) = \frac{1}{M} \sum_{m=0}^{M-1} f_e^*(m) g_e(x+m)$$

- 2-D ;

$$f_e(x,y) \circ g_e(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e^*(m,n) g_e(x+m, y+n)$$

- correlation theorem ;

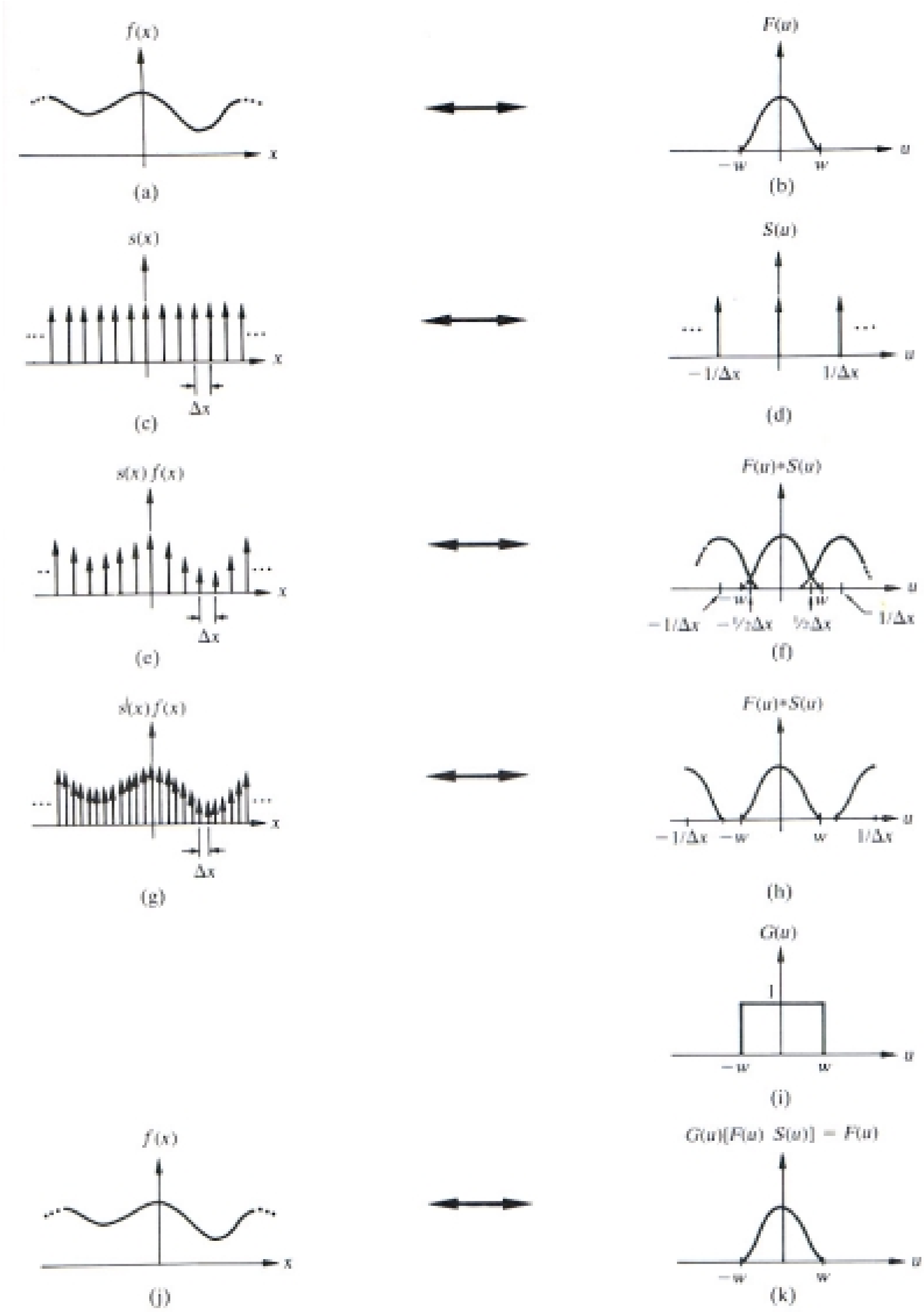
$$f(x,y) \circ g(x,y) \leftrightarrow F^*(u,v) G(u,v)$$

$$f(x,y) \circ g(x,y) \leftrightarrow F^*(u,v) G(u,v)$$

3.3.9 Sampling

1) 1-D functions

   - infinite sampling interval ;



- band-limited function

- sampling function s(x) ; comb function (or impulse train)
  : a train of impulse Δx units apart

- sampling of f(x) : s(x)f(x), F(u)*S(u)
  : convolution theorem

- transform : periodic with period $1/\Delta x$, overlap (aliasing)

- Nyquist rate :

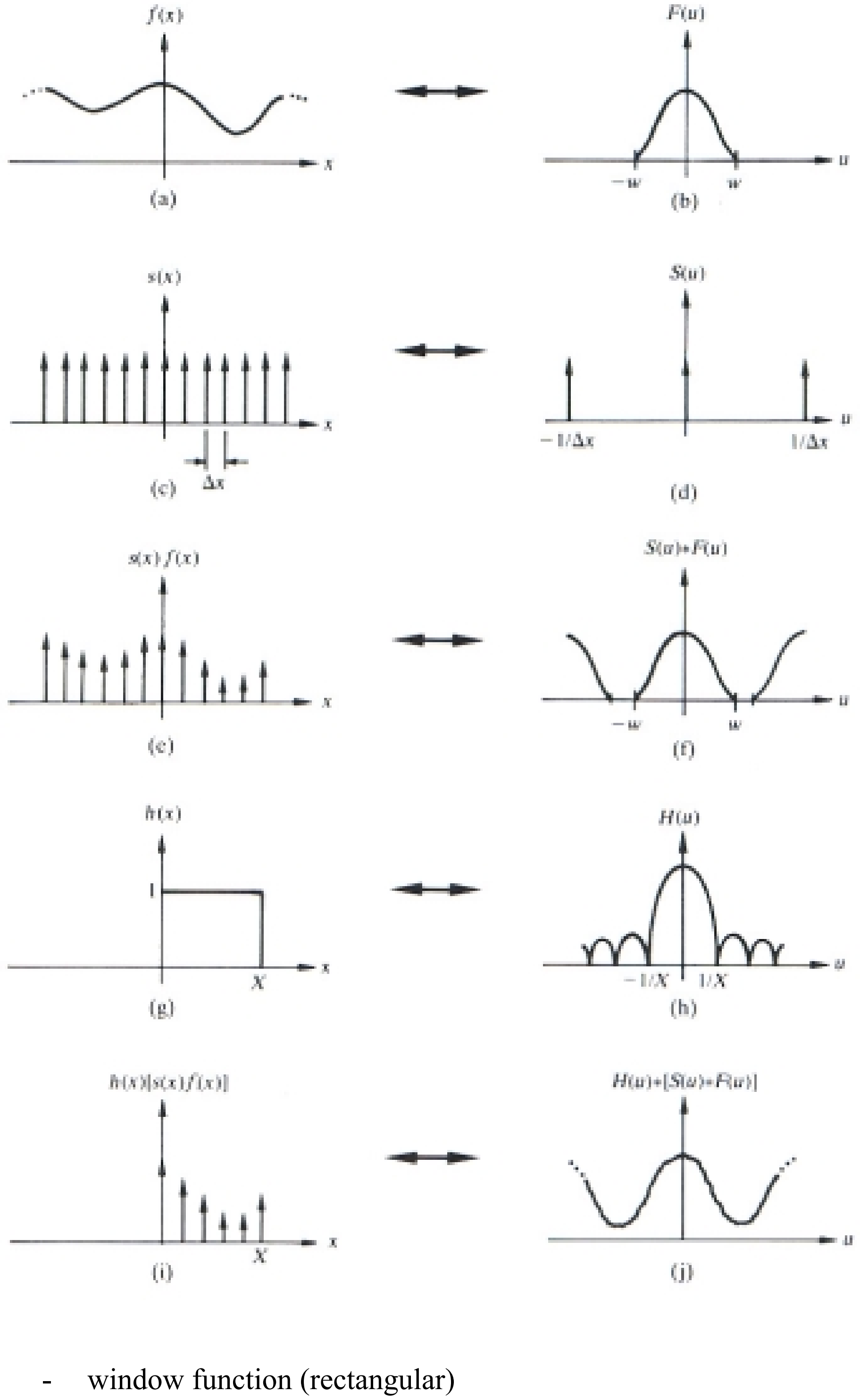
  $\Delta x \le \frac{1}{2W}$ or $\frac{1}{\Delta x} \ge 2W$

- Whittaker-Shannon sampling theorem :

  : complete recovery of a band-limited function from samples

- Finite sampling interval (practical case) :

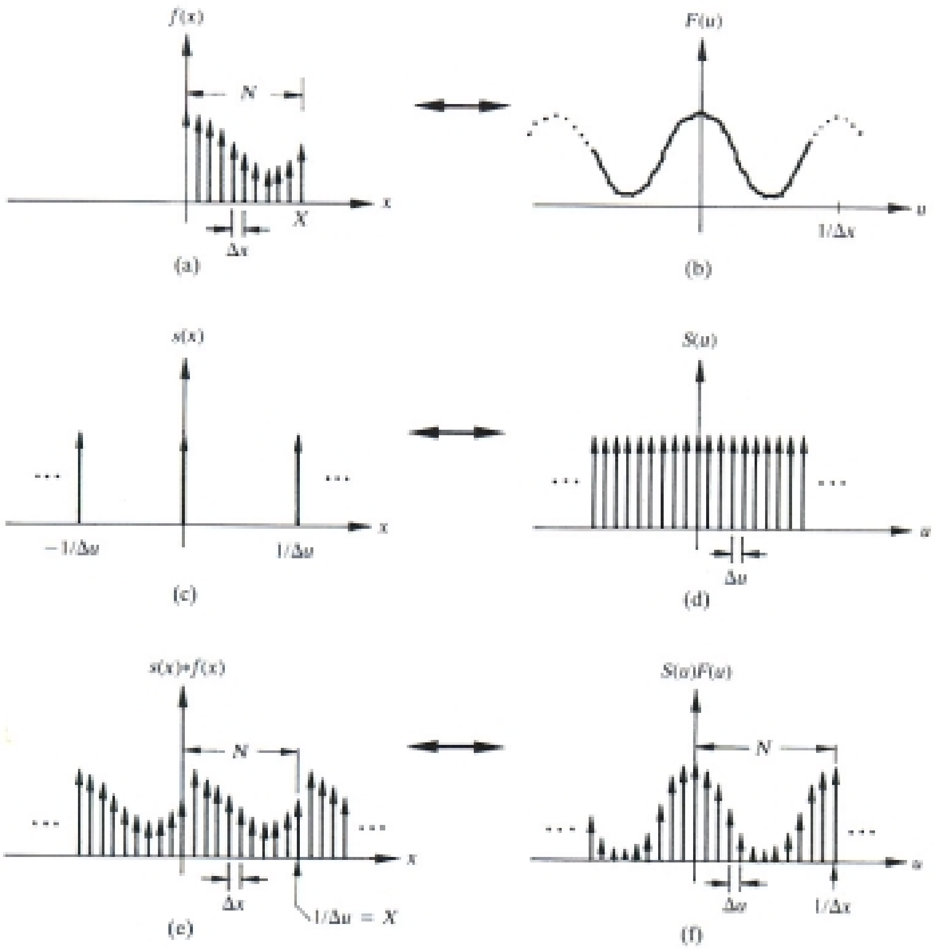
  : sampling interval: [0, X]



- window function (rectangular)

$$h(x) = \begin{cases} 1 & 0 \le x \le \omega \quad \text{:finite sampling interval} \\ 0 & elsewhere \end{cases}$$

- The periodicity of DFT :



  - N : samples of f(x) and F(u)

    → in x Domain ;

    $N\Delta x = X$

    → in freq. Domain ;

    $N\Delta u = 1/\Delta x \rightarrow \Delta u = 1/N\Delta x$

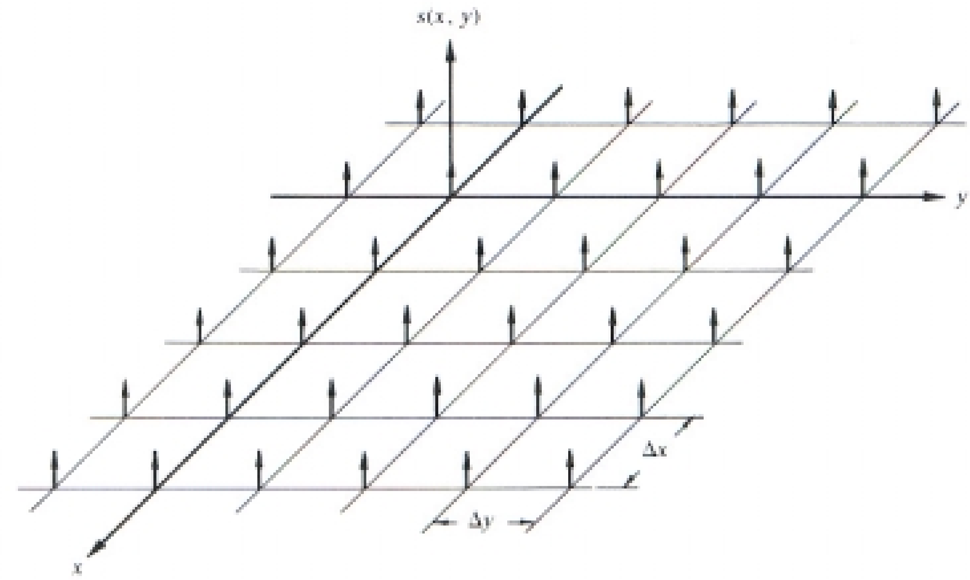
2) 2-D functions

- sampling process for 2-D functions :

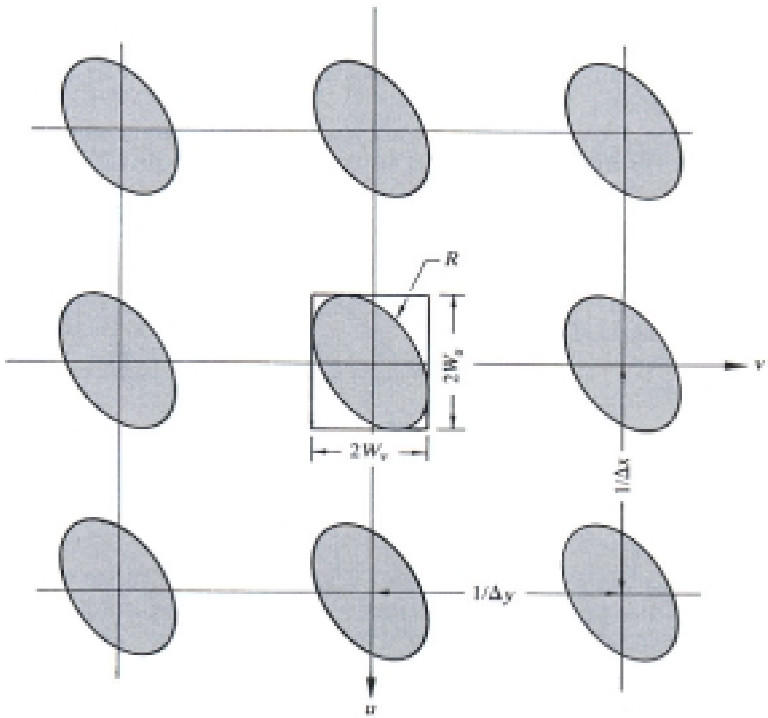
  - 2-D impulse function : δ(x,y) defined as ;

$$\iint_{-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0)dxdy = f(x_0, y_0)$$

2-D sampling function : s(x,y)



- sampled function : s(x,y)f(x,y)
- freq. Domain



- $S(u,v) * F(u,v)$

  where S(u,v) : a train of impulses with separation $1/\Delta x$ and $1/\Delta y$ in the u

and

  v directions

- f(x,y) : band limited
- $S(u,v) * F(u,v)$: periodic in two dimensions

- no aliasing condition :

  - if $2W_u$, $2W_v$: widths in the u and v directions

    $\rightarrow \frac{1}{\Delta x} \geq 2W_u$, $\frac{1}{\Delta y} \geq 2W_v$

- low-pass filtering :

$$G(u,v) = \begin{cases} 1 & -W_u \leq u \leq W_u \quad -W_v \leq u \leq W_v \\ 0 & otherwise \end{cases}$$

- 2-D DFT :

  - for N×N images

    $\Delta u = \frac{1}{N\Delta x}$, $\Delta v = \frac{1}{N\Delta y}$

    → complete 2-D period will be covered by NxN uniformly spaced values in both spatial and frequency domain