3.5.3 Discrete Cosine Transform (DCT)

- **2N-point extended DFT**

- 1-D DCT pair :

  - $C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{(2x+1)u\pi}{2N}\right]$

  - inverse ;

    $f(x) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{(2x+1)u\pi}{2N}\right]$

    where $\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{for} \quad u = 0 \\ \sqrt{\frac{2}{N}} & \text{for} \quad u = 1,2,\Lambda\ , N-1 \end{cases}$

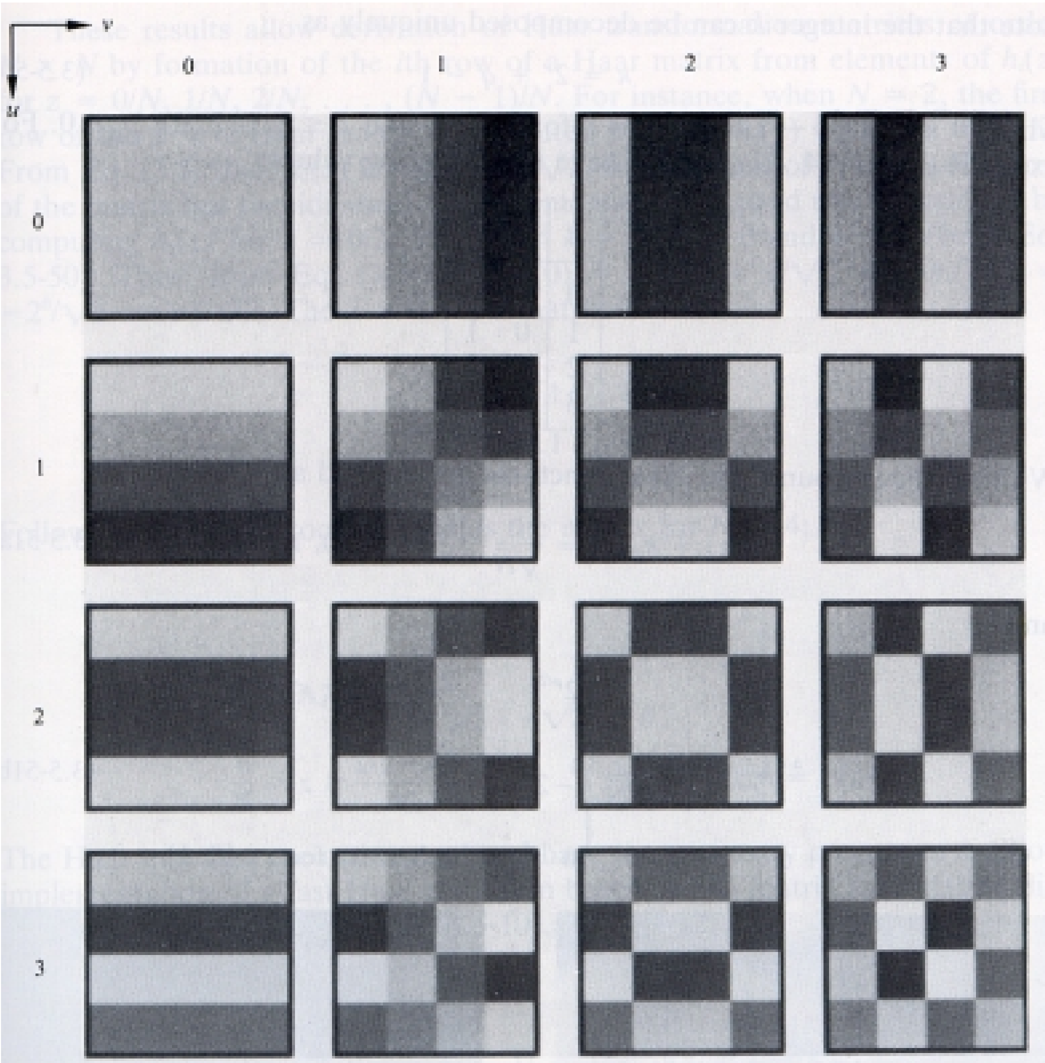
- 2-D DCT pair :

  - $C(u,v) = \alpha(u)\alpha(v) \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\left[\frac{(2x+1)u\pi}{2N}\right] \cos\left[\frac{(2y+1)v\pi}{2N}\right]$

    $f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \alpha(u)\alpha(v) C(u,v) \cos\left[\frac{(2x+1)u\pi}{2N}\right] \cos\left[\frac{(2y+1)v\pi}{2N}\right]$

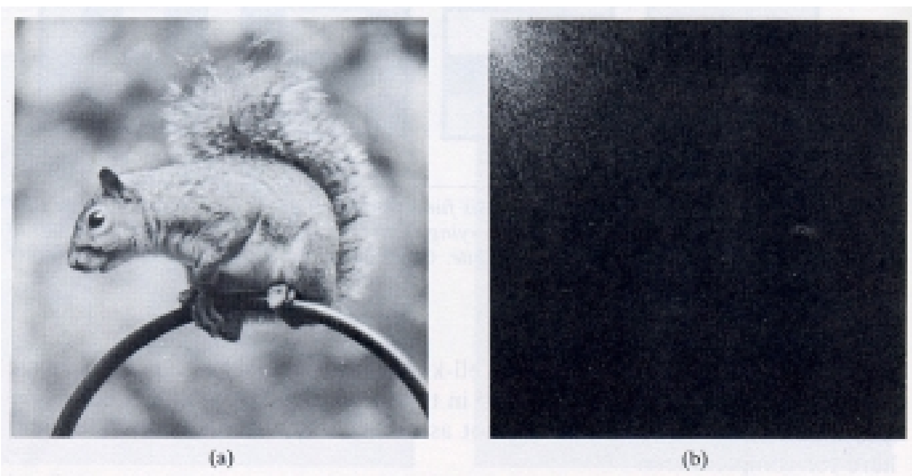
    : mainly used in image compression (excellent energy compaction)

- DCT Basis Images (N=4) :



DCT basis function for N=4

- DCTed image (log. Mag.)



3.5.4 Harr Transform

- **Based on Haar functions which are defined over continuous, closed interval**

- 1^{st} step in generating Haar Transform : (integer k decomposed uniquely)
  k=0,…, N-1, $N=2^n$

$$k = 2^p + q - 1 \tag{3.5-50}$$

where $0 \le p \le n - 1$, $q = 0$ or 1 for $p = 0$, and $1 \le q \le 2^p$ for $p \ne 0$. For example, if $N = 4$, k, q, and p have the following values:

k	p	q
0	0	0
1	0	1
2	1	1
3	1	2

With this background, the Haar functions are defined as

$$h_0(z) \triangleq h_{00}(z) = \frac{1}{\sqrt{N}} \quad \text{for } z \in [0, 1] \tag{3.5-51a}$$

and

$$h_k(z) \triangleq h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & \frac{q-1}{2^p} \le z < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \le z < \frac{q}{2^p} \\ 0 & \text{otherwise for } z \in [0, 1]. \end{cases} \tag{3.5-51b}$$

3.6 The Hotelling Transform (Karhunen-Loeve tranform)

: based on statistical properties of vector representation

- a random vector :

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  - mean vector ;

$$\mathbf{m}_x = E\{\mathbf{x}\}$$

  - covariance matrix ;

$$\mathbf{C}_x = E\{(\mathbf{x} - \mathbf{m}_x)(\mathbf{x} - \mathbf{m}_x)^T\}$$

    : n×n matrix

    ✓ element c_{ii} : the variance of x_i

    ✓ element c_{ij} : covariance between x_i and x_j

    ✓ $\mathbf{C}_x$: real and symmetric ($c_{ij} = c_{ji}$)

    ✓ if x_i and x_j are uncorrelated

    $\rightarrow$ $c_{ij} = c_{ji} = 0$

- for M vector samples, approximately

$$\mathbf{m}_x = \frac{1}{M}\sum_{k=1}^{M}\mathbf{x_k}$$

$$\mathbf{C}_x = \frac{1}{M}\sum_{k=1}^{M}\mathbf{x_k}\mathbf{x_k^T} - \mathbf{m}_x\mathbf{m_x^T}$$

- $\mathbf{C}_x$: real, symmetric :

  $\rightarrow$ a set of n orthonormal eigenvectors : always exist

  - $\mathbf{e}_i$: eigenvectors

- λ_i : eigenvalues i=1,2,…,n
  $\lambda_i \geq \lambda_{i+1}$
- $\mathbf{C}_x \mathbf{e}_i = \lambda_i \mathbf{e}_i$

- **A** :
  - ✓ Matrix formed from the eigenvectors $\mathbf{e}_i$
  - ✓ the first row of **A**
    : eigenvectors corresponding to the largest eigenvalue
  - ✓ the last row of **A**
    : eigenvectors corresponding to the smallest eigenvalue

- Hotelling transform :

$$\mathbf{y} = \mathbf{A}(\mathbf{x} - \mathbf{m})$$

Transformation matrix

- mean of $\mathbf{y}$: $0 \rightarrow \mathbf{m_y} = 0$

- $\mathbf{C_y} = \mathbf{A C_x A^T}$; "similarity transformation"

$$= \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

  - ✓ element of $\mathbf{y}$ vectors : uncorrelated

- $\mathbf{C_x}, \mathbf{C_y}$: the same eigenvalues, eigenvectors

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} + \mathbf{m_x}$$

ex) rotation of object (eigenvector 를 이용)

→ principal axes (or eigenvectors) 에 대하여 rotation

- rotation of axis;

$$y_1 = x_1 \cos\theta + x_2 \sin\theta$$

$$y_2 = -x_1 \sin\theta + x_2 \cos\theta$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- for p pixels ;

$$\mathbf{m}_x \cong \frac{1}{p} \sum_{i=1}^{p} \mathbf{x}_i$$

$$\mathbf{C}_x \cong \frac{1}{p} \left[\sum_{i=1}^{p} \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} \right] - \mathbf{m}_x \mathbf{m}_x^{\mathrm{T}}$$

where $\mathbf{x}_i$: position (a,b) of pixels

$\mathbf{x}_i, \mathbf{m}_x$: 2×1 vectors

$\mathbf{C}_x$: 2×1 matrix

- eigenvectors of $\mathbf{C}_x$: $\mathbf{e}_1, \mathbf{e}_2$
  - ✓ direction : max variance

⇒ the same as direction of eigenvectors ;

$$e_1 = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}, \qquad e_2 = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \mathbf{y} = \mathbf{A}(x - \mathbf{m}_x)$$

$$\text{where } \mathbf{A} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

*$\mathbf{m}_x$ 를 빼는것은 좌표축의 중심을 물체 영상의 중심에 놓는 역할을 한다.

- Reconstruction of x from y :

  - $\mathbf{x} = \mathbf{A}^{\mathbf{T}}\mathbf{y} + \mathbf{m}_x$
  - $\mathbf{A}_k$: k×n matrix from the k eigenvectors corresponding to the k largest eigenvalues
  - vector reconstructed by $\mathbf{A}_k$;

    $$\hat{\mathbf{x}} = \mathbf{A}_{\mathbf{k}}\mathbf{y} + \mathbf{m}_x$$

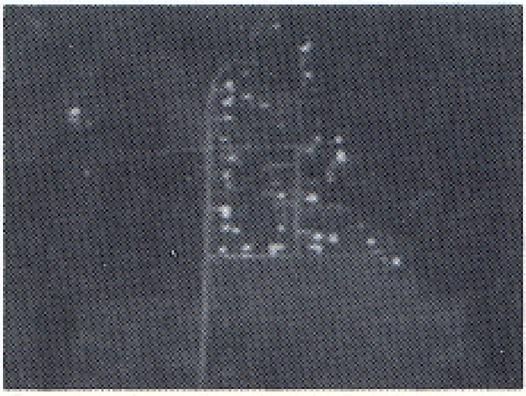
  - mean square error between $\mathbf{x}$ and $\hat{\mathbf{x}}$;

    $$e_{ms} = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{k} \lambda_j = \sum_{j=k+1}^{n} \lambda_j$$

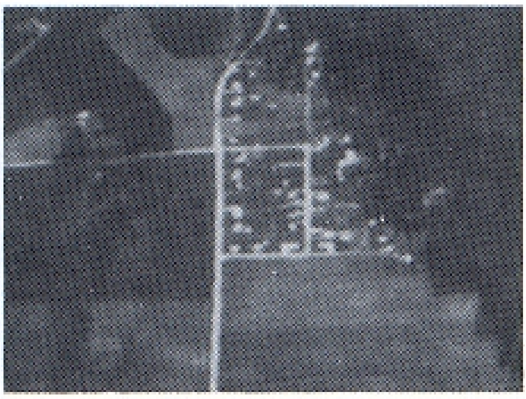
    → "Hotelling Transform"is optimal in the sense thatit minimizes the mean squared error

ex.) data compression (eigenvalue 를 이용)

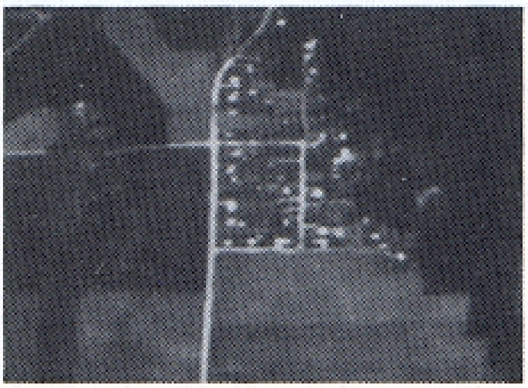
- 6 spectral images ; from 6-band multispectral scanner



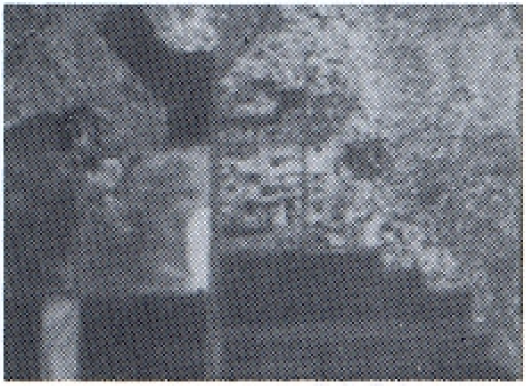
Channel 1



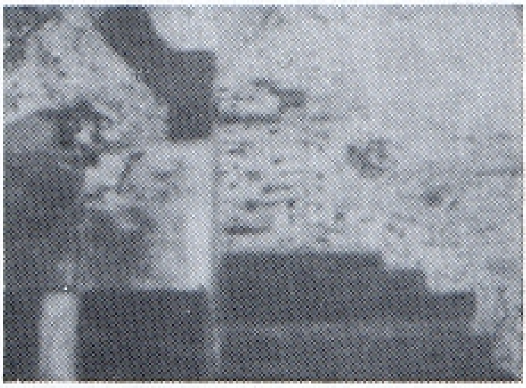
Channel 2



Channel 3

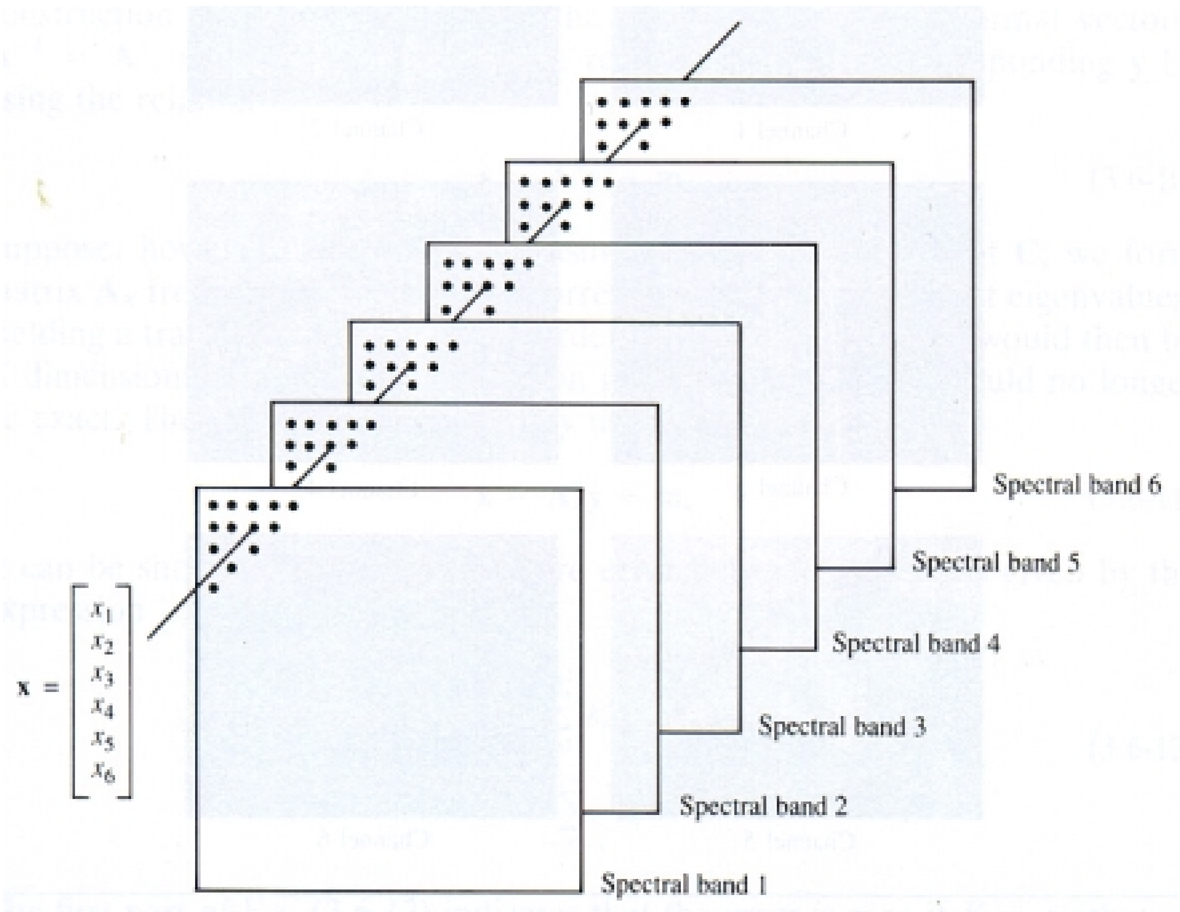


Channel 4



Channel 5

Channel 6



x =
x_1
x_2
x_3
x_4
x_5
x_6
Spectral band 6
Spectral band 5
Spectral band 4
Spectral band 3
Spectral band 2
Spectral band 1