5.4 Inverse Filtering

5.4.1 Formulation

- assume M=N :

  $\hat{\mathbf{f}} = \mathbf{H}^{-1}\mathbf{g} = (\mathbf{WDW}^{-1})^{-1}\mathbf{g}$

  $= \mathbf{WD}^{-1}\mathbf{W}^{-1}\mathbf{g}$

  $\mathbf{W}^{-1}\hat{\mathbf{f}} = \mathbf{D}^{-1}\mathbf{W}^{-1}\mathbf{g}$

  - Fourier Transform:

  $\hat{F}(u,v) = \dfrac{G(u,v)}{H(u,v)}$ for u, v = 0,1,2,…,N-1

  - $H(u,v)$ is considered as a filter

  ->: inverse filter method

- the restored image

  $\hat{f}(x,y) = \mathbf{F}^{-1}[\hat{F}(u,v)]$

  $= \mathbf{F}^{-1}[G(u,v)/H(u,v)]$

  for x, y = 0, 1, 2,…, N-1

  : normally implemented by FFT algorithm

- zeros of H(u,v)

  $\hat{f}(x,y)$ can not be sloved.

  → if zeros are known, it can be neglected in the computation of $\hat{F}(u,v)$

- In the presence of noise

  $\hat{F}(u,v) = F(u,v) + \dfrac{N(u,v)}{H(u,v)}$

  - if $H(u,v)$: zero or very small

$\rightarrow \quad \frac{N}{H}$ error term is dominant

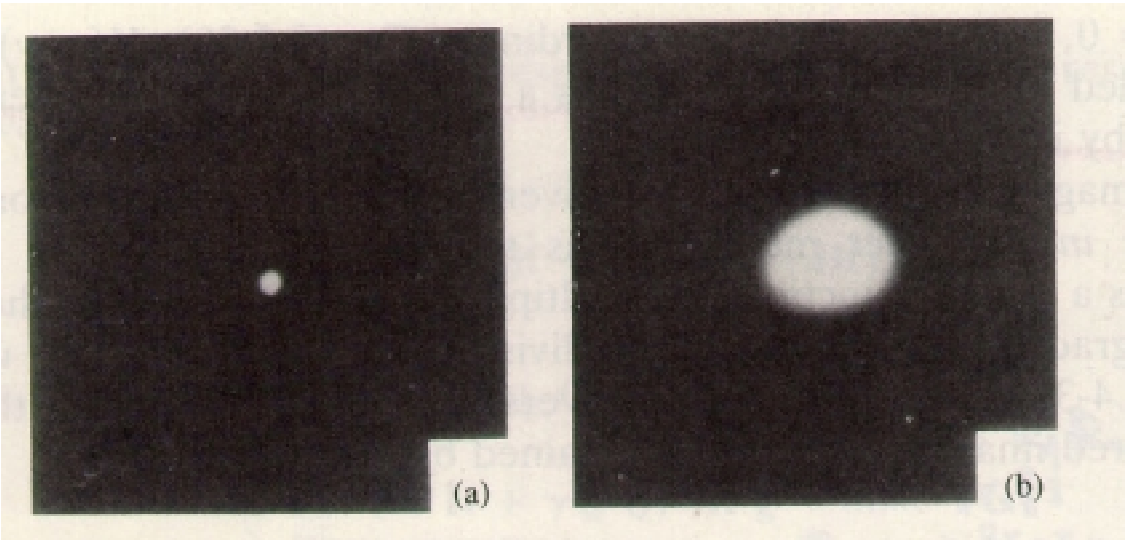
- in practice

  $H(u,v)$: drops off rapidly

  $N(u,v)$: drops off slowly

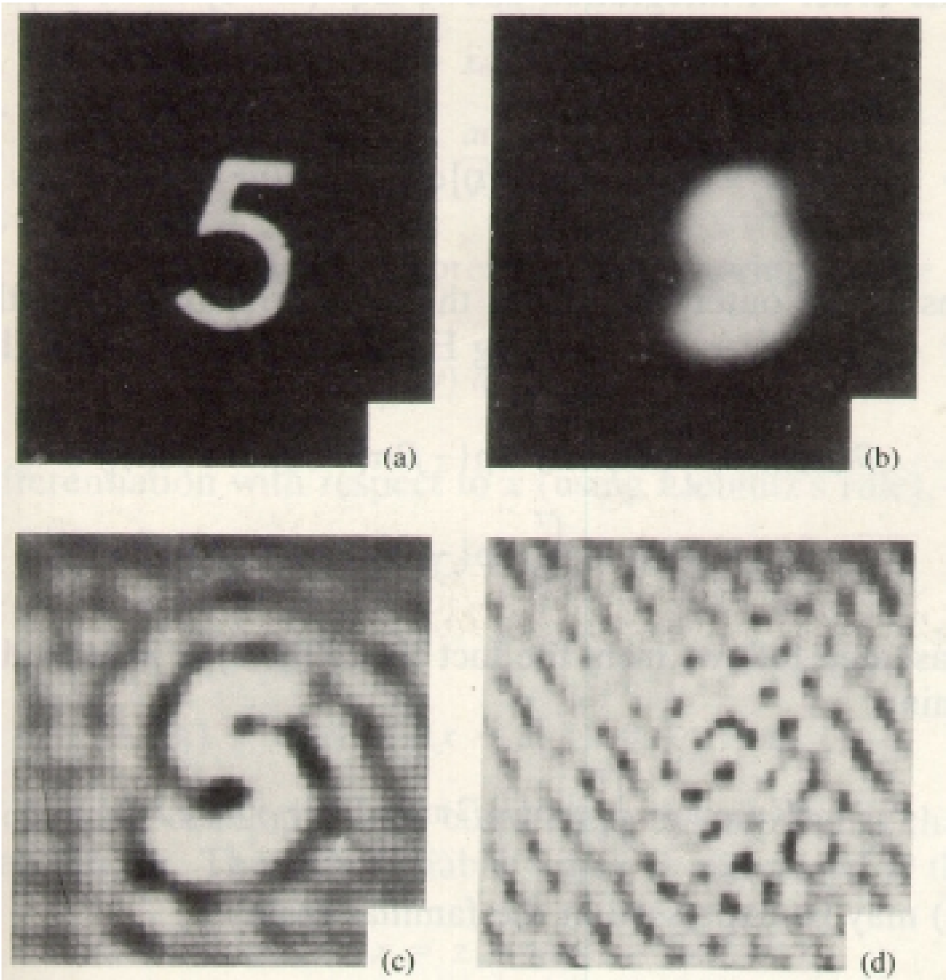
- reasonable result can be obtained by carrying out the restoration in a limited neighborhood about the origin in order to avoid small values of $H(u,v)$ -> Pseudo inverse filter

ex.)



(a) a point image : $f(x,y) = \delta(x,y)$

a image degraded by H: $G(u,v) = H(u,v)F(u,v) \approx H(u,v)$



(a) original image, (b) image blurred by $H(u,v)$, (c) restored image (not include excessively small values of $H(u,v)$), (d) restored image (using a larger neighborhood)

- If $H(u,v)$, $G(u,v)$, $N(u,v)$: all known

  An exact inverse filtering

  $$F(u,v) = \frac{G(u,v)}{H(u,v)} - \frac{N(u,v)}{H(u,v)}$$

  but $N(u,v)$ is seldom known

5.5 Least Mean Square (Wiener) Filter

- define
  - correlation matrices of **f** and **m**

    $\mathbf{R_f} = E\{\mathbf{ff}^T\}$

    $\mathbf{R_n} = E\{\mathbf{nn}^T\}$

    where E{ } : expected value operation

  - ij-th element of $\mathbf{R_f}$

    $E\{f_i f_j\}$: correlation between i-th and j-th elements of **f**

  - elements of **f** and **m** : real

    $\rightarrow\ E\{f_i f_j\} = E\{f_j f_i\}$, $E\{n_i n_j\} = E\{n_j n_i\}$

    $\Rightarrow\ \mathbf{R_f}$, $\mathbf{R_n}$: real symmetric matrices

  - for most image function
    correlation matrix
    - ✓ a band of nonzero elements about the main diagonal
    - ✓ zeros in the right upper and left lower corner regions
    - ✓ correlation between two pixels : a function of distance between the pixels and not their position $\rightarrow$ $\mathbf{R_f}$, $\mathbf{R_n}$: ~block circulant matrix
    - ✓
  - diagonalization

    $\mathbf{R_f} = \mathbf{WAW^{-1}}$

    $\mathbf{R_n} = \mathbf{WBW^{-1}}$

    where element of $\mathbf{A,B}$: Fourier transform of correlation element in $\mathbf{R_f}$, $\mathbf{R_n}$
  - FT of correlation

    Power spectrums (or spectral density) of $f_e(x,y), \eta_e(x,y)$ are $S_f(u,v), S_n(u,v)$,

respectively

- Define :

$$Q^T Q = \mathbf{R}_\mathbf{f}^{-1}\mathbf{R}_\mathbf{n}$$

$$\hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H} + \gamma\mathbf{R}_\mathbf{f}^{-1}\mathbf{R}_\mathbf{n})^{-1}\mathbf{H}^T\mathbf{g}$$

$$= (\mathbf{W}\mathbf{D}^*\mathbf{D}\mathbf{W}^{-1} + \gamma\mathbf{W}\mathbf{A}^{-1}\mathbf{B}\mathbf{W}^{-1})^{-1}\mathbf{W}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}$$

$$= \mathbf{W}(\mathbf{D}^*\mathbf{D} + \gamma\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{W}^{-1}\mathbf{W}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}$$

$$\mathbf{W}^{-1}\hat{\mathbf{f}} = (\mathbf{D}^*\mathbf{D} + \gamma\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{D}^*\mathbf{W}^{-1}\mathbf{g}$$

in terms of FT

$$\rightarrow \hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma\left[S_\eta(u,v)/S_f(u,v)\right]}\right]G(u,v)$$

$$= \left[\frac{1}{H(u,v)}\frac{|H(u,v)|^2}{(H(u,v))^2 + \gamma\left[S_\eta(u,v)/S_f(u,v)\right]}\right]G(u,v)$$

for u, v = 0, 1, 2,…, N-1, assume M=N

where $|H(u,v)|^2 = H^*(u,v)H(u,v)$

$\rightarrow$ **parametric Wiener filter**

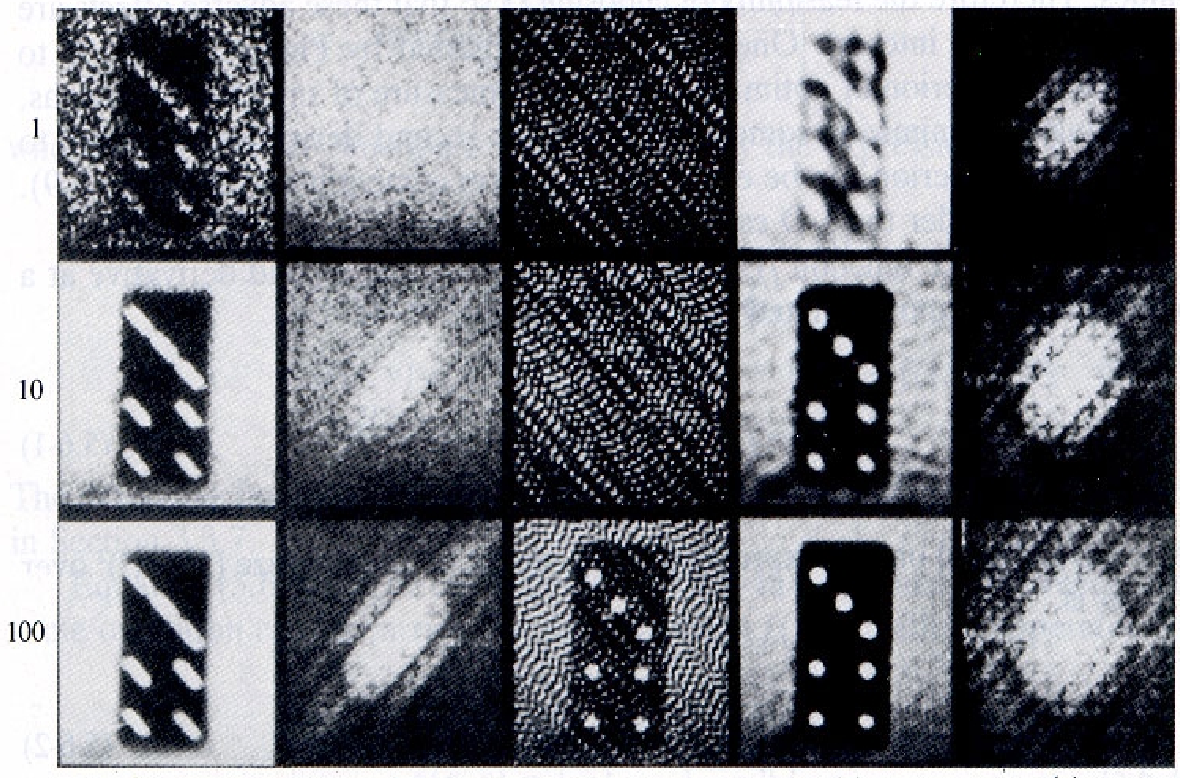
- when $\gamma = 1$ **Wiener filter**
- in the absence of noise, $S_\eta(u,v) = 0$

  $\rightarrow$ Wiener filter $\rightarrow$ ideal inverse filter
- when $\gamma = 1$, not optimal
- when $S_\eta(u,v)$, $S_f(u,v)$: unknown

$$\hat{F}(u,v) \approx \frac{1}{H(u,v)}\frac{|H(u,v)|^2}{|H(u,v)|^2 + k}G(u,v)$$

where k : constant

ex.)



(a) (b) (c) (d) (e)

(a) domino image

- linear motion (at - 45°): SNR : 1, 10, 100 times along column

(b) FT of (a)

(c) Images obtained by direct inverse filtering

(d) Images obtained by Wiener filter with $k = 2\sigma^2$

Where σ^2: variance of noise

(e) FT of (d)

5.6 constrained Least Squares Restoration

- Choosing **Q**

A criterion of smoothness: minimizing the second derivative

- second derivative of f(x)

$$\frac{\partial^2 f(x)}{\partial x^2} \approx f(x+1) - 2f(x) + f(x-1)$$

to min. $\left(\partial^2 f \middle/ \partial^2 x \right)^2$ over x

$$\text{min.} \left\{ \sum_x [f(x+1) - 2f(x) + f(x-1)] \right\}$$

- in matrix form

min. $\{\mathbf{f}^T \mathbf{C}^T \mathbf{C} \mathbf{f}\}$

where $\mathbf{C^T} = \begin{bmatrix} 1 & & & & & & \\ -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \\ & & & & & & 1 \end{bmatrix}$

: smoothing matrix (circulant matrix)

- in 2-D case

$$\text{min.} \left\{ \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \right\}^2$$; Laplacian Operator

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \approx 4f(x,y) - [f(x+1,y) - f(x-1,y) + f(x,y+1) + f(x,y-1)]$$

operator $p(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

extension

$$p_e(x,y) = \begin{cases} p(x,y) & 0 \le x \le 2 \quad and \quad 0 \le y \le 2 \\ 0 & 3 \le x \le M-1 \quad or \quad 3 \le y \le N-1 \end{cases}$$

$$f(x,y) : A \times B \rightarrow M \ge A+3-2,\ N \ge B+3-1$$

convolution

$$g_e(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e(m,n)\, p_e(x-m, y-n)$$

the smoothing criterion in matrix form

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_{M-1} & \mathbf{C}_{M-2} & \Lambda & \mathbf{C}_1 \\ \mathbf{C}_1 & \mathbf{C}_0 & \mathbf{C}_{M-1} & \Lambda & \mathbf{C}_2 \\ \mathbf{C}_2 & \mathbf{C}_1 & \mathbf{C}_0 & \Lambda & \mathbf{C}_3 \\ \mathrm{M} & & & & \\ \mathbf{C}_{M-1} & \mathbf{C}_{M-2} & \mathbf{C}_{M-3} & \Lambda & \mathbf{C}_0 \end{bmatrix}$$

where each submatrix $\mathbf{C}_j : N \times N$

$$\mathbf{C}_j = \begin{bmatrix} p_e(j,0) & p_e(j,N-1) & \Lambda & p_e(j,1) \\ p_e(j,1) & p_e(j,0) & \Lambda & p_e(j,2) \\ \mathrm{M} & & & \\ p_e(j,N-1) & p_e(j,N-2) & \Lambda & p_e(j,0) \end{bmatrix}$$

since $\mathbf{C}$: block-circulant matrix

→ $\mathbf{E} = \mathbf{W}^{-1}\mathbf{C}\mathbf{W}$

where $\mathbf{E}$: diagonal matrix whose elements are

$$\mathbf{E}(k,i) = \begin{cases} P\left(\left[\frac{k}{N}\right], k \bmod N\right) & if\ i = k \\ 0 & if\ i \ne k \end{cases}$$

P(u,v) : 2-D FT of $p_e(x,y)$

- min. $\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right\}^2 \rightarrow$ min. $\{\mathbf{f}^{-1}\mathbf{C}^{-1}\mathbf{C}\mathbf{f}\}$

where $\mathbf{f} : MN$, $\mathbf{C} : MN \times MN$

by letting $\mathbf{Q} = \mathbf{C}$

because $\|\mathbf{Qf}\|^2 = (\mathbf{Qf}^{T}(\mathbf{Qf} = \mathbf{f}^T\mathbf{Q}^T\mathbf{Qf}$

→ min. $\|\mathbf{Qf}\|^2$

- optimal solution

$$\hat{\mathbf{f}} = (\quad ^{T} \quad + \gamma\mathbf{C}^{T}\mathbf{C})^{-1} \quad ^{T}\mathbf{g}$$

$$= (\quad ^{*} \quad ^{-1} + \gamma \quad ^{*} \quad ^{-1})^{-1} \quad ^{*} \quad ^{-1}\mathbf{g}$$

$$\mathbf{W}^{-1}\hat{\mathbf{f}} = (\mathbf{D}^{*}\mathbf{D} + \gamma\mathbf{E}^{*}\mathbf{E})^{-1}\mathbf{D}^{*}\mathbf{W}^{-1}\mathbf{g}$$

in terms of FT

$$\hat{F}(u,v) = \left[\frac{H^{*}(u,v)}{|H(u,v)|^{2} + \gamma |P(u,v)|^{2}} \right] G(u,v)$$

- Choosing r

Define a residual vector $\mathbf{r}$ as

$$\mathbf{r} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}}.$$

$$\mathbf{r} = \mathbf{g} - \mathbf{H}(\mathbf{H}^{T}\mathbf{H} + \gamma\mathbf{C}^{T}\mathbf{C})^{-1}\mathbf{H}^{T}\mathbf{g}.$$

$$\begin{aligned} \phi(\gamma) &= \mathbf{r}^{T}\mathbf{r} \\ &= \|\mathbf{r}\|^{2} \end{aligned}$$

is a monotonically increasing function of γ.

$$\|\mathbf{r}\|^{2} = \|\mathbf{n}\|^{2} \pm a,$$

One simple approach is to

(1) specify an initial value of γ;
(2) compute $\hat{\mathbf{f}}$ and $\|\mathbf{r}\|^{2}$; and
(3) stop is satisfied; otherwise return to step 2 after increasing γ if $\|\mathbf{r}\|^{2} < \|\mathbf{n}\|^{2} - a$ or decreasing γ if $\|\mathbf{r}\|^{2} > \|\mathbf{n}\|^{2} + a$.