A Geometrical Analysis of a Class of Nonconvex Conic Programs for Convex Conic Reformulations of Quadratic and Polynomial Optimization Problems

Sunyoung Kim[*] Masakazu Kojima[†] Kim-Chuan Toh[‡]

January 8, 2019

Abstract

We present a geometrical analysis on the completely positive programming reformulation of quadratic optimization problems and its extension to polynomial optimization problems with a class of geometrically defined nonconvex conic programs and their covexification. The class of nonconvex conic programs is described with a linear objective function in a linear space $\mathbb{V}$, and the constraint set is represented geometrically as the intersection of a nonconvex cone $\mathbb{K} \subset \mathbb{V}$, a face $\mathbb{J}$ of the convex hull of $\mathbb{K}$ and a parallel translation $\mathbb{L}$ of a supporting hyperplane of the nonconvex cone $\mathbb{K}$. We show that under a moderate assumption, the original nonconvex conic program can equivalently be reformulated as a convex conic program by replacing the constraint set with the intersection of $\mathbb{J}$ and the hyperplane $\mathbb{L}$. The replacement procedure is applied to derive the completely positive programming reformulation of quadratic optimization problems and its extension to polynomial optimization problems.

Key words. Completely positive reformulation of quadratic and polynomial optimization problems, conic optimization problems, hierarchies of copositivity, faces of the completely positive cone.

AMS Classification. 90C20, 90C25, 90C26.

1 Introduction

Polynomial optimization problems (POPs) is a major class of optimization problems in theory and practice. Quadratic optimizations problems (QOPs) are, in particular, a

[*]Department of Mathematics, Ewha W. University, 52 Ewhayeodae-gil, Sudaemoon-gu, Seoul 03760, Korea (`skim@ewha.ac.kr`). The research was supported by 2017-R1A2B2005119.

[†]Department of Industrial and Systems Engineering, Chuo University, Tokyo 192-0393, Japan (`kojima@is.titech.ac.jp`). This research was supported by Grant-in-Aid for Scientific Research (A) 26242027.

[‡]Department of Mathematics, and Institute of Operations Research and Analytics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076 (`mattohkc@nus.edu.sg`). This research is supported in part by the Ministry of Education, Singapore, Academic Research Fund (Grant number: R-146-000-257-112).

widely studied subclass of POPs as they include many important NP-hard combinatorial problems such as binary QOPs, maximum stable set problems, graph partitioning problems and quadratic assignment problems. To numerically solve QOPs, a common approach is through solving their convex conic relaxations such as semidefinite programming relaxations [23, 21] and doubly nonnegative (DNN) relaxations [15, 19, 26, 28]. As those relaxations provide lower bounds of different qualities, the tightness of the lower bounds has been a very critical issue in assessing the strength of the relaxations. The completely positive programming (CPP) reformulation of QOPs, which provides their exact optimal values, has been extensively studied in theory. More specifically, QOPs over the standard simplex [9, 10], maximum stable set problems [12], graph partitioning problems [24], and quadratic assignment problems [25] are equivalently reformulated as CPPs. Burer's CPP reformulations [11] of a class of linearly constrained QOPs in nonnegative and binary variables provided a more general framework to study the specific problems mentioned above. See also the papers [1, 2, 8, 14, 22] for further developments.

Despite a great deal of studies on the CPP relaxation, its geometrical aspects have not been well understood. The main purpose of this paper is to present and analyze *essential features of the CPP reformulation of QOPs and its extension to POPs by investigating their geometry.* With the geometrical analysis, many existing equivalent reformulations of QOPs and POPs can be considered in a unified manner and deriving effective numerical methods for computing tight bounds can be facilitated. In particular, the class of QOPs that can be equivalently reformulated as CPPs in our framework includes Burer's class of linearly constrained QOPs in nonnegative and binary variables [11] as a special case; see Sections 2.2 and 6.1.

1.1 A geometric framework for the CPP relaxation of QOPs and its extension to POPs

A nonconvex conic optimization problem (COP), denoted as COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$, of the form presented below is the most distinctive feature of our framework for the CPP relaxation of QOPs and its extension to POPs. Let $\mathbb{V}$ be a finite dimensional vector space with the inner product $\langle \boldsymbol{A},\, \boldsymbol{B} \rangle$ for every pair of $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathbb{V}$. For a cone $\mathbb{K} \subset \mathbb{V}$, let co$\mathbb{K}$ denote the convex hull of $\mathbb{K}$ and $\mathbb{K}^*$ the dual of $\mathbb{K}$, *i.e.*, $\mathbb{K}^* = \{\boldsymbol{Y} \in \mathbb{V} : \langle \boldsymbol{X},\, \boldsymbol{Y} \rangle \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{K}\}$. Let $\boldsymbol{H}^0 \in \mathbb{V}$, which will be described more precisely in Section 2.2 for QOPs and in Section 5 for general POPs. For every cone $\mathbb{K}_0 \subset \mathbb{V}$ (not necessarily convex nor closed) and $\boldsymbol{Q}^0 \in \mathbb{V}$, we consider the COP given by

$$\text{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)\text{:}\ \zeta = \inf \left\{ \langle \boldsymbol{Q}^0,\, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K}_0,\ \langle \boldsymbol{H}^0,\, \boldsymbol{X} \rangle = 1 \right\}.$$

Although this problem takes a very simple form, it plays a fundamental role throughout. A key property is that COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$ is equivalent to its *covexification*, COP(co$\mathbb{K}_0, \boldsymbol{Q}^0$) under the following conditions (Theorem 3.2).

Condition I$_0$: COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$ is feasible and $\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{K}_0^*$.

Condition II$_0$: $\inf \left\{ \langle \boldsymbol{Q}^0,\, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K}_0,\ \langle \boldsymbol{H}^0,\, \boldsymbol{X} \rangle = 0 \right\} \geq 0$.

The only restrictive and essential condition among the conditions is $\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{K}_0^*$, while the others are natural. It means that $\{\boldsymbol{X} \in \mathbb{V} : \langle \boldsymbol{H}^0,\ \boldsymbol{X}\rangle = 0\}$ forms a supporting hyperplane of the cone $\mathbb{K}_0$ and that the feasible region of COP$(\mathbb{K}_0, \boldsymbol{Q}_0)$ is described as the intersection of the nonconvex cone $\mathbb{K}_0 \subset \mathbb{V}$ and a parallel translation of the supporting hyperplane of $\mathbb{K}_0$. Condition II$_0$ is necessary to ensure that the optimal value of COP$(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ is finite. See Figure 1 in Section 3.1 for illustrative examples of COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$ which satisfies Condition I$_0$ and II$_0$.

We consider a specific COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$ with $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$ for some cone $\mathbb{K} \subset \mathbb{V}$ and some face $\mathbb{J}$ of co$\mathbb{K}$. Note that $\mathbb{K} \cap \mathbb{J}$ is a nonconvex cone. Since $\mathbb{J}$ is a face of co$\mathbb{K}$, we have that co$(\mathbb{K} \cap \mathbb{J}) = \mathbb{J}$ ((i) of Lemma 3.4). It follows that the equivalence of COP$(\text{co}(\mathbb{K} \cap \mathbb{J}), \boldsymbol{Q}^0)$ and COP$(\mathbb{J}, \boldsymbol{Q}^0)$ holds trivially. This is another distinctive feature of our geometric framework.

In this paper, we mainly deal with a class of general POPs of the form:

$$\zeta^* = \inf \left\{ f_0(\boldsymbol{w}) : \boldsymbol{w} \in \mathbb{R}^n_+,\ f_i(\boldsymbol{w}) = 0\ (i = 1, \ldots, m) \right\}, \tag{1}$$

where $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional Euclidean space $\mathbb{R}^n$ and $f_i(\boldsymbol{w})$ a real valued polynomial function in $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ $(i = 0, \ldots, m)$. When all $f_i(\boldsymbol{w})$ $(i = 0, \ldots, m)$ are quadratic functions, (1) becomes a class of QOPs considered in this paper.

The equivalence of COP$(\text{co}(\mathbb{K} \cap \mathbb{J}), \boldsymbol{Q}^0)$ and COP$(\mathbb{J}, \boldsymbol{Q}^0)$ shown above can be applied to POP (1) by just reducing POP (1) to the form of COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$. This reduction is demonstrated in Section 2.2 for QOP cases, and in Section 5 for general POP cases. For the resulting COP$(\text{co}(\mathbb{K} \cap \mathbb{J}), \boldsymbol{Q}^0)$ to satisfy Conditions I$_0$ and II$_0$ with $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$, certain assumptions must be imposed. For example, if the feasible region of POP (1) is nonempty and bounded, and $f_i(\boldsymbol{w})$ $(i = 1, \ldots, m)$ are nonnegative for every $\boldsymbol{w} \geq \boldsymbol{0}$, COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ can be constructed such that Conditions I$_0$ and II$_0$ are satisfied with $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$ for some cone $\mathbb{K}$ and some face $\mathbb{J}$ of co$\mathbb{K}$. Consequently, COP$(\mathbb{J}, \boldsymbol{Q}^0)$ is indeed a convex COP reformulation of POP (1) with the same objective value $\zeta = \zeta^*$ (Theorem 5.2). Note that co$\mathbb{K}$ corresponds the CPP cone when POP (1) is a QOP, while it corresponds to an extension of the CPP cone for a general POP. Thus, $\mathbb{J}$ is a face of the CPP cone in the QOP case or a face of the extended CPP cone in the general POP case.

In the convexification from POP (1) to COP$(\mathbb{J}, \boldsymbol{Q}^0)$, the objective function $f_0(\boldsymbol{w})$ is relaxed to the linear function $\langle \boldsymbol{Q}^0,\ \boldsymbol{X}\rangle$ in $\boldsymbol{X} \in$ co$\mathbb{K}$. The problem COP$(\mathbb{J}, \boldsymbol{Q}^0)$, however, does not explicitly involve any linear equality in $\boldsymbol{X} \in$ co$\mathbb{K}$ induced from each equality constraint $f_i(\boldsymbol{w}) = 0$ $(i = 1, \ldots, m)$. In fact, the feasible region of COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ is geometrically represented in terms of a nonconvex cone $\mathbb{K}$, a face $\mathbb{J}$ of co$\mathbb{K}$ and a hyperplane $\{\boldsymbol{X} \in \mathbb{V} : \langle \boldsymbol{H}^0,\ \boldsymbol{X}\rangle = 1\}$. This formulation is essential to derive the convex COP reformulation COP$(\mathbb{J}, \boldsymbol{Q}^0)$ of QOPs and POPs in a simple geometric setting.

1.2 Relations to existing works

The geometric framework mentioned in the previous section generalizes the authors' previous work [1, 2, 3, 4, 19]. A convex reformulation of a nonconvex COP in a vector space $\mathbb{V}$ was also discussed and the results obtained there were applied to QOPs in

[1, 3, 19] and POPs in [2, 4]. Unlike the current framework, a fundamental difference in the previous framework lies in utilizing a nonconvex COP of the form

$$\zeta = \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle : \begin{array}{l} \boldsymbol{X} \in \mathbb{K}, \ \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1, \\ \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \ (p = 1, \ldots, m) \end{array} \right\}, \tag{2}$$

where $\mathbb{K} \subset \mathbb{V}$ denotes a cone, $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 0, \ldots, m)$ and $\boldsymbol{H}^0 \in \mathbb{V}$. In [3, 4, 19], they imposed the assumption that $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 0, \ldots, m)$ in addition to $\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{K}^*$ and a condition similar to Condition II_0. Under this assumption,

$$\mathbb{J} = \{ \boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \ (p = 1, \ldots, m) \} \tag{3}$$

forms a face of $\mathrm{co}\mathbb{K}$ (Lemma 2.1). However, the converse is not true. A face $\mathbb{J}$ of $\mathrm{co}\mathbb{K}$ can be represented as in (3) by some $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \ldots, m)$ iff it is an exposed face of $\mathrm{co}\mathbb{K}$; hence if $\mathbb{J}$ is a non-exposed face of $\mathrm{co}\mathbb{K}$, such a representation in terms of some $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \ldots, m)$ is impossible. Very recently, Zhang [29] showed that the CPP cone with dimension not less than 5 is not facially exposed, i.e., some of its faces are non-exposed (see also [6, 13] for geometric properties of the CPP cone). Thus, our framework using $\mathrm{COP}(\mathrm{co}(\mathbb{K} \cap \mathbb{J}), \boldsymbol{Q}^0)$ is more general than the work using (2) in [1, 2, 3, 4, 19].

The class of QOPs that can be reformulated as equivalent CPPs of the form $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$ in our framework covers most of the known classes of QOPs that can be reformulated as CPPs mentioned above, including Burer's class [11] of linearly constrained QOPs in nonnegative and binary variables. With respect to extensions to POPs presented in [2, 4, 22], our geometric framework using $\mathrm{COP}((\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ can be regarded as a generalization of the framework proposed in [2, 4] where a class of POPs of the form (1) is reduced to COP (2). In [22], Peña, Vera and Zuluaga introduced the cone of completely positive tensor as an extension of the completely positive matrix for deriving equivalent convex relaxation of POPs. The class of POPs that can be convexified using their completely positive tensor cone is similar to our class that can be reformulated as equivalent CPPs of the form $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$. In fact, one of the two conditions imposed on their class, (i) of Theorem 4 in [22], corresponds to our condition (30), which was originated from a hierarchy of copositivity condition proposed in [1]. The other condition using "the horizon cone" in (ii) of Theorem 4 of [22], is different from our condition (31), but they are similar in nature (see Section 6 of [1]). We should mention, however, that our framework is quite different form theirs.

The above discussions show the versatility of our geometric framework in that it is applicable to almost all known equivalent reformulations of QOPs as well as the more general case of POPs.

1.3 Outline of the paper

After introducing some notation and symbols in Section 2.1, we present how a general QOP can be reduced to $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$ in Section 2.2, and present some fundamental properties of cones and their faces in Section 2.3. We establish the equivalence of $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}_0)$ and its convexification $\mathrm{COP}(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ under Conditions I_0 and II_0 in Section 3.1, and derive the equivalence of $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}_0)$ and its convexification $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$ by taking

$\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$ for some cone $\mathbb{K} \subset \mathbb{V}$ and some face $\mathbb{J}$ of co$\mathbb{K}$ in Section 3.2. In Section 4.1, we introduce a hierarchy of copositivity condition to represent a face $\mathbb{J}$ of the convex hull co$\mathbb{K}$ of a cone $\mathbb{K} \subset \mathbb{V}$ as in (3). This connects two forms of a nonconvex COP, COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and COP (2). In Section 4.2, some sufficient conditions for $\boldsymbol{Q}^p$ $(p = 1, \ldots, m)$ to represent a face $\mathbb{J}$ of co$\mathbb{K}$ as in (3) are provided. Section 5 discusses convex COP reformulations of POPs as applications of the results obtained in Sections 3 and 4. We discuss homogenizing polynomials and an extension of the completely positive cone in Sections 5.1 and 5.2, respectively. We then construct a convex COP reformulation of POP (1) in Sections 5.3 and 5.4. In Section 6, we illustrate how we can apply the main theorems established in Section 5.4 to QOPs and POPs through examples. Finally, we conclude the paper in Section 7.

2 Preliminaries

2.1 Notation and symbols

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space consisting of column vectors $\boldsymbol{w} = (w_1, \ldots, w_n)$, $\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$, $\mathbb{S}^n$ the linear space of $n \times n$ symmetric matrices with the inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij}$, and $\mathbb{S}^n_+$ the cone of positive semidefinite matrices in $\mathbb{S}^n$. $\mathbb{Z}^n$ denotes the set of integer column vectors in $\mathbb{R}^n$, and $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$. $\boldsymbol{c}^T$ denotes the transposition of a column vector $\boldsymbol{c} \in \mathbb{R}^n$. When $\mathbb{R}^{1+n}$ is used, the first coordinate of $\mathbb{R}^{1+n}$ is indexed by 0 and $\boldsymbol{x} \in \mathbb{R}^{1+n}$ is written as $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$ with $\boldsymbol{w} \in \mathbb{R}^n$. Also each matrix $\boldsymbol{X} \in \mathbb{S}^{1+n} \subset \mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$ has elements X_{ij} $(i = 0, \ldots, n, j = 0, \ldots, n)$.

Let $\mathbb{V}$ be a finite dimensional linear space with the inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle$ for every pair of $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathbb{V}$ and $\|\boldsymbol{A}\| = \langle \boldsymbol{A}, \boldsymbol{A} \rangle^{1/2}$ for every $\boldsymbol{A}$ in $\mathbb{V}$. We say that $\mathbb{K} \subset \mathbb{V}$ is a *cone*, which is not necessarily convex nor closed, if $\lambda \boldsymbol{A} \in \mathbb{K}$ for every $\boldsymbol{A} \in \mathbb{K}$ and $\lambda \geq 0$. Let co$\mathbb{K}$ denote the convex hull of a cone $\mathbb{K}$, and cl$\mathbb{K}$ the closure of $\mathbb{K}$. $\mathbb{S}^{1+n}$ may be regarded as a special case of a linear space $\mathbb{V}$ in the subsequent discussions. Since $\mathbb{K}$ is a cone, we see that $\text{co}\mathbb{K} = \left\{ \sum_{p=1}^m \boldsymbol{X}^p : \boldsymbol{X}^p \in \mathbb{K} \ (p = 1, \ldots, m) \text{ for some } m \in \mathbb{Z}_+ \right\}$. The *dual* of a cone $\mathbb{K}$ is defined as $\mathbb{K}^* = \{\boldsymbol{Y} \in \mathbb{V} : \langle \boldsymbol{Y}, \boldsymbol{X} \rangle \geq 0 \text{ for every } \ \boldsymbol{X} \in \mathbb{K}\}$. From the definition, we know that $\mathbb{K}^* = (\text{co}\mathbb{K})^*$. It is well-known and also easily proved by the separation theorem of convex sets that $\mathbb{K}^{**} = \text{cl co}\mathbb{K}$, the closure of co$\mathbb{K}$.

We note that a cone $\mathbb{K}$ is convex iff $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}^i \in \mathbb{K}$ whenever $\boldsymbol{X}^i \in \mathbb{K}$ $(i = 1, \ldots, m)$. Let $\mathbb{K}$ be a convex cone in a linear space $\mathbb{V}$. A convex cone $\mathbb{J} \subset \mathbb{K}$ is said to be a *face* of $\mathbb{K}$ if $\boldsymbol{X}^1 \in \mathbb{J}$ and $\boldsymbol{X}^2 \in \mathbb{J}$ whenever $\boldsymbol{X} = \boldsymbol{X}^1/2 + \boldsymbol{X}^2/2 \in \mathbb{J}$, $\boldsymbol{X}^1 \in \mathbb{K}$ and $\boldsymbol{X}^2 \in \mathbb{K}$ (the standard definition of a face of a convex set), or, if $\boldsymbol{X}^i \in \mathbb{J}$ $(i = 1, \ldots, m)$ whenever $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}^i \in \mathbb{J}$ and $\boldsymbol{X}^i \in \mathbb{K}$ $(i = 1, \ldots, m)$ (the equivalent characterization of a face of a convex cone). The equivalence can be easily shown by induction. A face $\mathbb{J}$ of $\mathbb{K}$ is *proper* if $\mathbb{J} \neq \mathbb{K}$, and a proper face $\mathbb{J}$ of $\mathbb{K}$ is *exposed* if there is a nonzero $\boldsymbol{P} \in \mathbb{K}^*$ such that $\mathbb{J} = \{\boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{P}, \boldsymbol{X} \rangle = 0\}$. A proper face of $\mathbb{K}$ is *non-exposed*, if it is not exposed. In general, if $\mathcal{T}(\mathbb{J})$ denotes the tangent linear space of a face $\mathbb{J}$ of $\mathbb{K}$, then $\mathbb{J} = \mathbb{K} \cap \mathcal{T}(\mathbb{J})$. Here the *tangent linear space* $\mathcal{T}(\mathbb{J})$ of a face $\mathbb{J}$ of $\mathbb{K}$ is defined as the smallest linear subspace of $\mathbb{V}$ that contains $\mathbb{J}$. The *dimension of a face* $\mathbb{J}$ is defined as the dimension of its tangent linear subspace $\mathcal{T}(\mathbb{J})$. We say that $\boldsymbol{P} \in \mathbb{V}$ is *copositive* on a cone $\mathbb{K} \subset \mathbb{V}$ if $\langle \boldsymbol{P}, \boldsymbol{X} \rangle \geq 0$ for every $\boldsymbol{X} \in \mathbb{K}$, *i.e.*, $\boldsymbol{P} \in \mathbb{K}^*$.

Let $\boldsymbol{H}^0 \in \mathbb{V}$. For every $\mathbb{K} \subset \mathbb{V}$ and $\rho \geq 0$, let $G(\mathbb{K}, \rho) = \left\{ \boldsymbol{X} \in \mathbb{K} : \langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = \rho \right\}$. In addition, given any $\boldsymbol{P} \in \mathbb{V}$, we consider the following conic optimization problem

$$\mathrm{COP}(\mathbb{K}, \boldsymbol{P}, \rho): \quad \zeta(\mathbb{K}, \boldsymbol{P}, \rho) = \inf \left\{ \langle \boldsymbol{P}, \boldsymbol{X} \rangle : \boldsymbol{X} \in G(\mathbb{K}, \rho) \right\}.$$

Note that we use the convention that $\zeta(\mathbb{K}, \boldsymbol{P}, \rho) = +\infty$ if $G(\mathbb{K}, \rho) = \emptyset$, and that $\mathrm{COP}(\mathbb{K}, \boldsymbol{P}, 1)$ coincides with $\mathrm{COP}(\mathbb{K}, \boldsymbol{P})$ introduced in Section 1. In the subsequent sections, we often use $\zeta(\mathbb{K}, \boldsymbol{P}, \rho)$ with $\rho \geq 0$, but $\mathrm{COP}(\mathbb{K}, \boldsymbol{P}, \rho)$ only for $\rho = 1$. For simplicity, we use the notation $\mathrm{COP}(\mathbb{K}, \boldsymbol{P})$ for $\mathrm{COP}(\mathbb{K}, \boldsymbol{P}, 1)$.

2.2 A class of QOPs with linear equality, complementarity and binary constraints in nonnegative variables

In this section, we first consider Burer's class of QOPs which were shown to be equivalent to their CPP reformulations under mild assumptions (see (7) and (8) below) in [11]. For the reader who might be more familiar with QOPs than POPs, our purpose here is to show how our geometrical analysis works for QOPs, before presenting the rigorous derivation of our convexification procedure for the POP (1).

Let $\boldsymbol{C} \in \mathbb{S}^n$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{\ell \times n}$, $\boldsymbol{b} \in \mathbb{R}^\ell$, $I_{\mathrm{bin}} \subset \{1, \ldots, n\}$ (the index set for binary variables) and $I_{\mathrm{comp}} \subset \{(j,k) : 1 \leq j < k \leq n\}$ (the index set for pairs of complementary variables). For simplicity of notation, we assume that $I_{\mathrm{bin}} = \{1, \ldots, q\}$ for some $q \geq 0$; if $q = 0$ then $I_{\mathrm{bin}} = \emptyset$. Consider a QOP of the following form:

$$\zeta_{\mathrm{QOP}} \;=\; \inf \left\{ \boldsymbol{w}^T \boldsymbol{C} \boldsymbol{w} + 2\boldsymbol{c}^T \boldsymbol{w} : \begin{array}{l} \boldsymbol{w} \in \mathbb{R}^n_+, \\ f_1(\boldsymbol{w}) \equiv (\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b})^T (\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}) = \boldsymbol{0}, \\ f_2(\boldsymbol{w}) \equiv \sum_{(j,k) \in I_{\mathrm{comp}}} w_j w_k = 0, \\ f_{p+2}(\boldsymbol{w}) \equiv w_p(1 - w_p) = 0 \ (p = 1, \ldots, q) \end{array} \right\}. \quad (4)$$

Assume that the feasible region of QOP (4) is nonempty. Note that the multiple complementarity constraints $w_j w_k = 0$ $((j,k) \in I_{\mathrm{comp}})$ in $\boldsymbol{w} \in \mathbb{R}^n_+$ is written as the single equality constraint $f_2(\boldsymbol{w}) = 0$ in $\boldsymbol{w} \in \mathbb{R}^n_+$ mainly for simplicity.

Let

$$\begin{aligned} \boldsymbol{\Gamma}^{1+n} &= \left\{ \boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^{1+n}_+ \right\}, \ \mathbb{CPP}^{1+n} = \mathrm{co}\boldsymbol{\Gamma}^{1+n}, \\ \mathbb{COP}^{1+n} &= (\mathbb{CPP}^{1+n})^* = \left\{ \boldsymbol{Y} \in \mathbb{S}^{1+n} : \boldsymbol{x}^T \boldsymbol{Y} \boldsymbol{x} \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{R}^{1+n}_+ \right\}. \end{aligned}$$

Then, $\boldsymbol{\Gamma}^{1+n}$ forms a nonconvex cone in $\mathbb{S}^{1+n}$. The convex cones $\mathbb{CPP}^{1+n}$ and $\mathbb{COP}^{1+n}$ are known as the *completely positive cone* and the *copositive cone* in the literature [7], respectively. We know that

$$\boldsymbol{\Gamma}^{1+n} \subset \mathbb{CPP}^{1+n} \subset \mathbb{S}^{1+n}_+ \cap \mathbb{N}^{1+n} \subset \mathbb{S}^{1+n}_+ \subset \mathbb{S}^{1+n}_+ + \mathbb{N}^{1+n} \subset \mathbb{COP}^{1+n} = (\boldsymbol{\Gamma}^{1+n})^*,$$

where $\mathbb{S}^{1+n}_+$ denotes the cone of positive semidefinite matrices in $\mathbb{S}^{1+n}$, and $\mathbb{N}^{1+n}$ the cone of matrices with all nonnegative elements in $\mathbb{S}^{1+n}$. The cone $\mathbb{S}^{1+n}_+ \cap \mathbb{N}^{1+n}$ is often called the *doubly nonnegative (DNN) cone*.

We now transform QOP (4) to COP($\mathbf{\Gamma}^{1+n} \cap \mathbb{J}, \boldsymbol{Q}^0$) for some convex cone $\mathbb{J} \subset \mathbb{CPP}^{1+n}$ and some $\boldsymbol{Q}^0 \in \mathbb{S}^{1+n}$. Let $m = q + 2$. We first introduce the following homogeneous quadratic functions in $(x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$:

$$\left.\begin{array}{l} \bar{f}_0(\boldsymbol{x}) = \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w} + 2x_0\boldsymbol{c}^T\boldsymbol{w}, \ \bar{f}_1(\boldsymbol{x}) = (\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}x_0)^T(\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}x_0), \\ \bar{f}_2(\boldsymbol{x}) = \sum_{(j,k)\in I_{\mathrm{comp}}} w_j w_k, \ \bar{f}_p(\boldsymbol{x}) = w_{p-2}(x_0 - w_{p-2}) \ (p = 3, \ldots, m). \end{array}\right\} \tag{5}$$

Then, we can rewrite QOP (4) as

$$\zeta_{\mathrm{QOP}} \ = \ \inf\left\{\bar{f}_0(\boldsymbol{x}) : \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}_+, \ x_0 = 1, \ \bar{f}_p(\boldsymbol{x}) = 0 \ (p = 1, \ldots, m)\right\}. \tag{6}$$

Since each $\bar{f}_p(\boldsymbol{x})$ is a homogeneous quadratic function in $\boldsymbol{x} \in \mathbb{R}^{1+n}$, it can be rewritten as $\bar{f}_p(\boldsymbol{x}) = \langle \boldsymbol{Q}^p, \ \boldsymbol{x}\boldsymbol{x}^T\rangle$ for some $\boldsymbol{Q}^p \in \mathbb{S}^{1+n}$ $(p = 1, \ldots, n)$. As a result, QOP (6) can be further transformed into

$$\zeta_{\mathrm{QOP}} \ = \ \inf\left\{\langle \boldsymbol{Q}^0, \ \boldsymbol{X}\rangle : \begin{array}{l} \boldsymbol{X} \in \mathbf{\Gamma}^{1+n}, \ \langle \boldsymbol{H}^0, \ \boldsymbol{X}\rangle = 1, \\ \langle \boldsymbol{Q}^p, \ \boldsymbol{X}\rangle = 0 \ (p = 1, \ldots, m) \end{array}\right\},$$

where $\boldsymbol{H}^0$ denotes the matrix in $\mathbb{S}^{1+n}$ with the $(0,0)$th element $H^0_{00} = 1$ and 0 elsewhere. We note that $\boldsymbol{x} = (1, \boldsymbol{w}) \in \mathbb{R}^{1+n}_+$ iff $\boldsymbol{x}\boldsymbol{x}^T \in \mathbf{\Gamma}$ and $\langle \boldsymbol{H}^0, \ \boldsymbol{X}\rangle = 1$. By considering the convex cone $\mathbb{J} = \left\{\boldsymbol{X} \in \mathbb{CPP}^{1+n} : \langle \boldsymbol{Q}^p, \ \boldsymbol{X}\rangle = 0 \ (p = 1, \ldots, m)\right\}$, we can rewrite the above problem as the COP

$$\begin{aligned} \zeta_{\mathrm{QOP}} \ &= \ \inf\left\{\langle \boldsymbol{Q}^0, \ \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbf{\Gamma}^{1+n} \cap \mathbb{J}, \ \langle \boldsymbol{H}^0, \ \boldsymbol{X}\rangle = 1\right\} \\ &= \ \inf\left\{\langle \boldsymbol{Q}^0, \ \boldsymbol{X}\rangle : \boldsymbol{X} \in G(\mathbf{\Gamma}^{1+n} \cap \mathbb{J}, 1)\right\} = \zeta(\mathbf{\Gamma}^{1+n} \cap \mathbb{J}, \boldsymbol{Q}^0, 1), \end{aligned}$$

which is equivalent to QOP (4). Thus, we have derived COP($\mathbf{\Gamma}^{1+n} \cap \mathbb{J}, Q^0$) with a convex cone $\mathbb{J} \subset \mathbb{CPP}^{1+n}$.

If Conditions I_0 and II_0 are satisfied with $\mathbb{K}_0 = \mathbf{\Gamma}^{1+n} \cap \mathbb{J}$, then COP($\mathbb{K}_0, \boldsymbol{Q}^0$) is equivalent to its covexification COP($\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0$), *i.e.*, $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0) = \zeta(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ (Theorem 3.2). If, in addition, $\mathbb{J}$ is a face of $\mathbb{CPP}^{1+n}$, then $\mathrm{co}\mathbb{K}_0 = \mathrm{co}(\mathbf{\Gamma}^{1+n} \cap \mathbb{J}) = \mathbb{J}$ (Lemma 3.4). Hence, COP($\mathbf{\Gamma}^{1+n} \cap \mathbb{J}, \boldsymbol{Q}^0$) is equivalent to its covexification COP($\mathbb{J}, \boldsymbol{Q}^0$), which forms a CPP reformulation of QOP (4) such that $\zeta(\mathbb{J}, \boldsymbol{Q}^0, 1) = \zeta_{\mathrm{QOP}}$.

In Burer [11], the following conditions are imposed on QOP (4) to derive its equivalent CPP reformulation:

$$w_i \leq 1 \text{ if } \boldsymbol{w} \in L \equiv \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{w} - \boldsymbol{b} = \boldsymbol{0}\right\} \text{ and } i \in I_{\mathrm{bin}}, \tag{7}$$

$$w_j = 0 \text{ and } w_k = 0 \text{ if } \boldsymbol{w} \in L_\infty \equiv \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}\right\} \text{ and } (j,k) \in I_{\mathrm{comp}}. \tag{8}$$

Although his CPP reformulation of QOP (4) is described quite differently from COP($\mathbb{J}, \boldsymbol{Q}^0$), the conditions (7) and (8) are sufficient not only for $\mathbb{J}$ to be a face of $\mathbb{CPP}^{1+n}$ but also for $\zeta(\mathbb{J}, \boldsymbol{Q}^0, 1) = \zeta_{\mathrm{QOP}}$ to hold. This fact will be shown in Section 6.1.

2.3 Fundamental properties of cones and their faces

The following lemma will play an essential role in the subsequent discussions.

Lemma 2.1. *Let $\mathbb{K} \subset \mathbb{V}$ be a cone. The following results hold.*

(i) $\mathbb{K}^* = (\mathrm{co}\mathbb{K})^*$.

(ii) Assume that $\boldsymbol{P} \in \mathbb{V}$ is copositive on $\mathbb{K}$. Then $\mathbb{J} = \{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{P}, \boldsymbol{X}\rangle = 0\}$ forms an exposed face of $\mathrm{co}\mathbb{K}$.

(iii) Let $\mathbb{J}_0 = \mathrm{co}\mathbb{K}$. Assume that $\mathbb{J}_p$ is a face of $\mathbb{J}_{p-1}$ $(p = 1, \ldots, m)$. Then $\mathbb{J}_\ell$ is a face of $\mathbb{J}_p$ $(0 \le p \le \ell \le m)$.

Proof. (i) $\mathbb{K}^* \supset (\mathrm{co}\mathbb{K})^*$ follows from the fact that $\mathbb{K} \subset \mathrm{co}\mathbb{K}$. To prove the converse inclusion, suppose that $\boldsymbol{X} \in \mathbb{K}^*$. Choose $\boldsymbol{Y} \in \mathrm{co}\mathbb{K}$ arbitrarily. Then there exist $\boldsymbol{Y}^i \in \mathbb{K}$ $(i = 1, \ldots, k)$ such that $\boldsymbol{Y} = \sum_{i=1}^k \boldsymbol{Y}^i$. Since $\boldsymbol{X} \in \mathbb{K}^*$ and $\boldsymbol{Y}^i \in \mathbb{K}$, we have that $\langle \boldsymbol{Y}^i, \boldsymbol{X}\rangle \ge 0$ $(i = 1, \ldots, k)$. It follows that $\langle \boldsymbol{Y}, \boldsymbol{X}\rangle = \sum_{i=1}^k \langle \boldsymbol{Y}^i, \boldsymbol{X}\rangle \ge 0$. Hence we have shown that $\langle \boldsymbol{Y}, \boldsymbol{X}\rangle \ge 0$ for every $\boldsymbol{Y} \in \mathrm{co}\mathbb{K}$. Therefore $\boldsymbol{X} \in \mathrm{co}\mathbb{K}^*$.

(ii) Let $\boldsymbol{X} = \boldsymbol{X}^1/2 + \boldsymbol{X}^2/2 \in \mathbb{J}$, $\boldsymbol{X}^1 \in \mathrm{co}\mathbb{K}$ and $\boldsymbol{X}^2 \in \mathrm{co}\mathbb{K}$. By the assumption, $\langle \boldsymbol{P}, \boldsymbol{X}^1\rangle \ge 0$ and $\langle \boldsymbol{P}, \boldsymbol{X}^2\rangle \ge 0$. From $\boldsymbol{X} = \boldsymbol{X}^1/2 + \boldsymbol{X}^2/2 \in \mathbb{J}$, we also see that $0 = \langle \boldsymbol{P}, \boldsymbol{X}\rangle = \langle \boldsymbol{P}, \boldsymbol{X}^1\rangle/2 + \langle \boldsymbol{P}, \boldsymbol{X}^2\rangle/2$. Hence $\langle \boldsymbol{P}, \boldsymbol{X}^1\rangle = \langle \boldsymbol{P}, \boldsymbol{X}^2\rangle = 0$. Therefore $\boldsymbol{X}^1 \in \mathbb{J}$ and $\boldsymbol{X}^2 \in \mathbb{J}$, and we have shown that $\mathbb{J}$ is a face of $\mathrm{co}\mathbb{K}$. Note that $\mathbb{J}$ is exposed by definition.

(iii) We only prove the case where $m = 2$ since the general case where $m \ge 3$ can be proved by induction. Let $\boldsymbol{X} = \boldsymbol{X}^1/2 + \boldsymbol{X}^2/2 \in \mathbb{J}_2$, $\boldsymbol{X}^1 \in \mathbb{J}_0$ and $\boldsymbol{X}^2 \in \mathbb{J}_0$. It follows from $\mathbb{J}_2 \subset \mathbb{J}_1$ that $\boldsymbol{X} \in \mathbb{J}_1$. Since $\mathbb{J}_1$ is a face of $\mathbb{J}_0$, we obtain that $\boldsymbol{X}^1 \in \mathbb{J}_1$ and $\boldsymbol{X}^2 \in \mathbb{J}_1$. Now, since $\mathbb{J}_2$ is a face of $\mathbb{J}_1$, $\boldsymbol{X}^1 \in \mathbb{J}_2$ and $\boldsymbol{X}^2 \in \mathbb{J}_2$ follow. Thus we have shown that $\mathbb{J}_2$ is a face of $\mathbb{J}_0$. □

3 Main results

Given a nonconvex cone $\mathbb{K}_0 \subset \mathbb{V}$, $\boldsymbol{H}^0 \in \mathbb{V}$ and $\boldsymbol{Q}^0 \in \mathbb{V}$, the problem $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$ minimizes the linear objective function $\langle \boldsymbol{Q}^0, \boldsymbol{X}\rangle$ over the nonconvex feasible region $G(\mathbb{K}_0, 1)$. In Section 2.2, we have derived such a nonconvex COP from QOP (4). We will also see in Section 5 that a general class of POPs can be reformulated as such a nonconvex COP. By replacing $\mathbb{K}_0$ with its convex hull $\mathrm{co}\mathbb{K}_0$, we obtain $\mathrm{COP}(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ that minimizes the same linear objective function over the convex feasible region $G(\mathrm{co}\mathbb{K}_0, 1)$. Hence $\mathrm{COP}(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ turns out to be a convex conic optimization problem. We call this process the *covexification* of $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$. Since $\mathbb{K}_0 \subset \mathrm{co}\mathbb{K}_0$, we have that $G(\mathrm{co}\mathbb{K}_0, 1) \supset G(\mathbb{K}_0, 1)$ and $\zeta(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) \le \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ hold in general. Hence $\zeta(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ provides a lower bound for the optimal value of the original QOP or POP from which $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$ is derived. If $\zeta(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$, we call $\mathrm{COP}(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ as a *convex COP reformulation* of $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$ (and the original QOP or POP). In this case, $\mathrm{COP}(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ solves the original QOP or POP in the sense that $\zeta(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ coincides with their optimal values. The main result of this section is the characterization of the convex COP reformulation of $\mathrm{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$.

Throughout this section, we fix a linear space $\mathbb{V}$ and $\boldsymbol{H}^0 \in \mathbb{V}$.

3.1 A simple conic optimization problem

For every $\mathbb{K}_0 \subset \mathbb{V}$ and $\boldsymbol{Q}^0 \in \mathbb{V}$, we consider COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$. To ensure $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = \zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ in Theorem 3.2, we will assume Conditions I_0 and II_0 introduced in Section 1. We note that the feasibility of COP$(\mathbb{K}_0, \boldsymbol{Q}_0)$ in Condition I_0 can be stated as $G(\mathbb{K}_0, 1) \neq \emptyset$, and Condition II_0 as $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0) \geq 0$.

Lemma 3.1 and Theorem 3.2 below may be regarded as special cases of Lemma 3.1 and Theorem 3.1 of [3]. Although Lemma 3.1 and Theorem 3.2 can be derived if $m = 0$ is used in [3], here we present their proofs for the paper to be self-contained.

Lemma 3.1. *Let $\mathbb{K}_0 \subset \mathbb{V}$ be a cone. Assume that Condition I_0 holds. Then,*

(i) $\text{co}G(\mathbb{K}_0, 0) = G(\text{co}\mathbb{K}_0, 0)$.

(ii) For every $\boldsymbol{P} \in \mathbb{V}$, $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = \zeta(\text{co}\mathbb{K}_0, \boldsymbol{P}, 0)$.

(iii) For every $\boldsymbol{P} \in \mathbb{V}$, $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = \begin{cases} 0 & \text{if } \zeta(\mathbb{K}_0, \boldsymbol{P}, 0) \geq 0 \text{ holds,} \\ -\infty & \text{otherwise.} \end{cases}$

Theorem 3.2. *Let $\mathbb{K}_0 \subset \mathbb{V}$ be a cone and $\boldsymbol{Q}^0 \in \mathbb{V}$. Assume that Condition I_0 holds. Then,*

(i) $G(\text{co}\mathbb{K}_0, 1) = \text{co}G(\mathbb{K}_0, 1) + \text{co}G(\mathbb{K}_0, 0)$.

(ii) $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) + \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0)$.

(iii) $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ *iff*

$$\text{Condition } II_0 \text{ or } \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = -\infty \text{ holds.} \tag{9}$$

Before presenting the proofs of Lemma 3.1 and Theorem 3.2, we show an illustrative example.

Example 3.3. Let $\mathbb{V} = \mathbb{R}^2$, $\boldsymbol{d}^1 = (4, 0)$, $\boldsymbol{d}^2 = (4, 2)$, $\boldsymbol{d}^3 = (-3, 3)$ and $\mathbb{K}_0 = \bigcup_{i=1}^3 \left\{\lambda \boldsymbol{d}^i : \lambda \geq 0\right\}$. We consider two cases (see (a) and (b) of Figure 1, respectively).

(a) Let $\boldsymbol{H}^0 = (0.5, 1)$, which lies in the interior of $\mathbb{K}_0^*$. In this case, we see that $G(\mathbb{K}_0, 1) = \{(-2, 2), (1, 0.5), (2, 0)\}$, $G(\text{co}\mathbb{K}_0, 1) = \text{co}G(\mathbb{K}_0, 1) =$ the line segment jointing $(-2, 2)$ and $(2, 0)$, and $G(\mathbb{K}_0, 0) = G(\text{co}\mathbb{K}_0, 0) = \{\boldsymbol{0}\}$. Hence $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{P}, 0) = \zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = 0$ for every $\boldsymbol{P} \in \mathbb{R}^2$ and Condition II_0 holds for every $\boldsymbol{Q}^0 \in \mathbb{R}^2$. Thus all assertions of Lemma 3.1 and Theorem 3.2 hold.

(b) Let $\boldsymbol{H}^0 = (0, 1)$, which lies in the boundary of $\mathbb{K}_0^*$. In this case, we see that $G(\mathbb{K}_0, 1) = \{(-1, 1), (2, 1)\}$, $G(\text{co}\mathbb{K}_0, 1) = \{(x_1, 1) : -1 \leq x_1\}$, and $G(\mathbb{K}_0, 0) = G(\text{co}\mathbb{K}_0, 0) = \{(x_1, 0) : 0 \leq x_1\}$. Hence (i) and (ii) of Lemma 3.1, and (i) of Theorem 3.2 follow. Take $\boldsymbol{Q}^0 = \boldsymbol{P} = (p_1, p_2) \in \mathbb{R}^2$ arbitrarily. If $p_1 \geq 0$ then $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = \zeta(\text{co}\mathbb{K}_0, \boldsymbol{P}, 0) = 0$, and both COP$(\mathbb{K}_0, \boldsymbol{P})$ and COP$(\text{co}\mathbb{K}_0, \boldsymbol{P})$ have a common optimal solution at $(-1, 1)$ with the optimal value $\zeta(\mathbb{K}_0, \boldsymbol{P}, 1) = \zeta(\text{co}\mathbb{K}_0, \boldsymbol{P}, 1) = -p_1 + p_2$; hence (iii) of Lemma 3.1, (ii) and (iii) of Theorem 3.2 hold. Now assume that $p_1 < 0$. Then we see that $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0) = \zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = \zeta(\text{co}\mathbb{K}_0, \boldsymbol{P}, 0) = -\infty$. This implies that (ii) of Lemma 3.1 holds, and that Condition II_0 is violated. We also see that $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = 2p_1 + p_2$. In this case, (iii) of Theorem 3.2 asserts that $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) \neq \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$. In fact, we have that $-\infty = \zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) < \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = 2p_1 + p_2$.

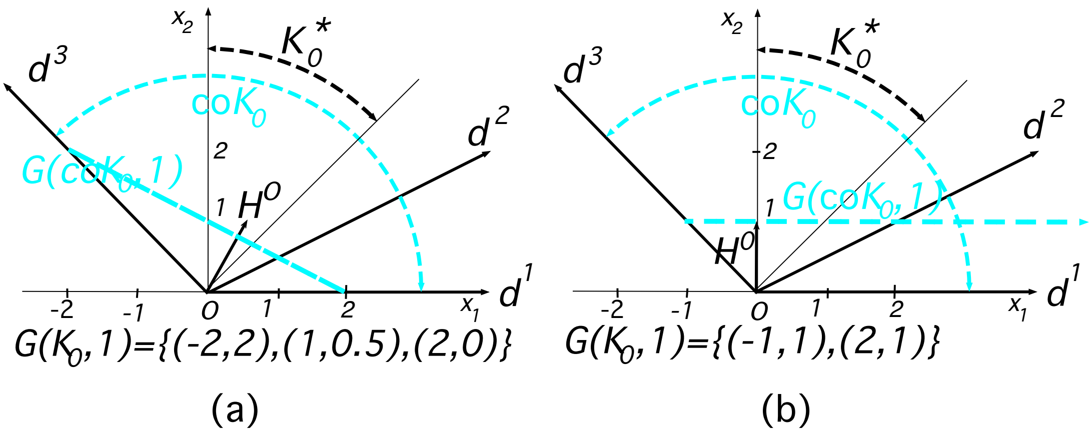

Figure 1: Illustration of COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$ and COP$(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0)$ under Conditions I_0 and II_0, where $\mathbb{V} = \mathbb{R}^2$, $\mathbb{K}_0 = \bigcup_{i=1}^3 \left\{\lambda \boldsymbol{d}^i : \lambda \geq 0\right\}$ and $G(\mathbb{K}_0, 1) = \{\boldsymbol{X} \in \mathbb{K}_0 : \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = 1\}$ (the feasible region of COP$(\mathbb{K}_0, \boldsymbol{Q}^0)$). In case (a) where we take $\boldsymbol{H}^0 = (0.5, 1) \in \mathbb{K}_0^*$, Condition I_0 and Condition II_0 are satisfied for any choice of $\boldsymbol{Q}^0 \in \mathbb{R}^2$. In case (b) where we take $\boldsymbol{H}^0 = (0, 1) \in \mathbb{K}_0^*$, Condition I_0 is satisfied, but Condition II_0 is satisfied iff the first coordinate Q_1^0 of $\boldsymbol{Q}^0 \in \mathbb{R}^2$ is nonnegative. See Example 3.3 for more details.

Proof of Lemma 3.1. (i) Since $G(\text{co}\mathbb{K}_0, 0)$ is a convex subset of $\mathbb{V}$ containing $G(\mathbb{K}_0, 0)$, we see that co$G(\mathbb{K}_0, 0) \subset G(\text{co}\mathbb{K}_0, 0)$. To show the converse inclusion, assume that $\boldsymbol{X} \in G(\text{co}\mathbb{K}_0, 0)$. Then there exist $\boldsymbol{X}^i \in \mathbb{K}_0$ $(i = 1, 2, \ldots, r)$ such that $\boldsymbol{X} = \sum_{i=1}^r \boldsymbol{X}^i$. By Condition I_0, we know that $\langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle \geq 0$ $(i = 1, 2, \ldots, r)$. Thus, each $\boldsymbol{X}^i$ satisfies $\boldsymbol{X}^i \in \mathbb{K}_0$ and $\langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle = 0$, or equivalently $\boldsymbol{X}^i \in G(\mathbb{K}_0, 0)$ $(i = 1, 2, \ldots, r)$. Therefore, $\boldsymbol{X} = \sum_{i=1}^r \lambda_i \boldsymbol{X}^i \in \text{co}G(\mathbb{K}_0, 0)$.

(ii) Let $\boldsymbol{P} \in \mathbb{V}$. We observe that

$$\begin{aligned}
\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) &= \inf\left\{\langle \boldsymbol{P}, \boldsymbol{X}\rangle : \boldsymbol{X} \in \text{co}G(\mathbb{K}_0, 0)\right\} \text{ (since } \langle \boldsymbol{P}, \boldsymbol{X}\rangle \text{ is linear in } \boldsymbol{X}) \\
&= \inf\left\{\langle \boldsymbol{P}, \boldsymbol{X}\rangle : \boldsymbol{X} \in G(\text{co}\mathbb{K}_0, 0)\right\} \text{ (by (i))} \\
&= \inf \zeta(\text{co}\mathbb{K}_0, \boldsymbol{P}, 0).
\end{aligned}$$

(iii) Since the objective function $\langle \boldsymbol{P}, \boldsymbol{X}\rangle$ in the description of $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0)$ is linear and its feasible region $G(\mathbb{K}_0, 0)$ forms a cone, we know that $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = 0$ or $-\infty$ and that $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) = 0$ iff the objective value is nonnegative for all feasible solutions, *i.e.*, $\zeta(\mathbb{K}_0, \boldsymbol{P}, 0) \geq 0$ holds. □

Proof of Theorem 3.2. (i) To show the inclusion $G(\text{co}\mathbb{K}_0, 1) \subset \text{co}G(\mathbb{K}_0, 1) + \text{co}G(\mathbb{K}_0, 0)$, assume that $\boldsymbol{X} \in G(\text{co}\mathbb{K}_0, 1)$. Then there exist $\boldsymbol{X}^i \in \mathbb{K}_0 \subset \text{co}\mathbb{K}_0$ $(i = 1, 2, \ldots, r)$ such that

$$\boldsymbol{X} = \sum_{i=1}^r \boldsymbol{X}^i \text{ and } \ 1 = \langle \boldsymbol{H}^0, \boldsymbol{X}\rangle = \sum_{i=1}^r \langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle.$$

By Condition I_0, $\langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle \geq 0$ $(i = 1, \ldots, r)$. Let

$$\begin{aligned}
& I_+ = \left\{ i : \langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle > 0 \right\}, \ I_0 = \left\{ j : \langle \boldsymbol{H}^0, \boldsymbol{X}^j\rangle = 0 \right\}, \\
& \mu_i = \langle \boldsymbol{H}^0, \boldsymbol{X}^i\rangle, \ \boldsymbol{Y}^i = (1/\mu_i)\boldsymbol{X}^i \ (i \in I_+), \ \boldsymbol{Y} = \sum_{i \in I_+} \boldsymbol{X}^i, \\
& \mu_j = 1/\left|I_0\right|, \ \boldsymbol{Z}^j = (1/\mu_j)\boldsymbol{X}^j \ \ (j \in I_0), \ \boldsymbol{Z} = \sum_{j \in I_0} \boldsymbol{X}^j,
\end{aligned}$$

where $|I_0|$ denotes the number of elements in I_0. Then $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$, and

$$\begin{aligned}
\mu_i &> 0, \ \boldsymbol{Y}^i \in \mathbb{K}_0, \ 1 = \langle \boldsymbol{H}^0, \boldsymbol{Y}^i\rangle \ (i \in I_+), \ 1 = \sum_{i \in I_+} \mu_i, \ \boldsymbol{Y} = \sum_{i \in I_+} \mu_i \boldsymbol{Y}_i, \\
\mu_j &> 0, \ \boldsymbol{Z}^j \in \mathbb{K}_0, \ 0 = \langle \boldsymbol{H}^0, \boldsymbol{Z}^j\rangle \ (j \in I_0), \ 1 = \sum_{j \in I_0} \mu_j, \ \boldsymbol{Z} = \sum_{j \in I_0} \mu_j \boldsymbol{Z}^j.
\end{aligned}$$

Thus, $\boldsymbol{Y}^i \in G(\mathbb{K}_0, 1)$ $(i \in I_+)$, $\boldsymbol{Z}^j \in G(\mathbb{K}_0, 0)$ $(j \in I_0)$, $\boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}_0, 1)$, $\boldsymbol{Z} \in \mathrm{co}G(\mathbb{K}_0, 0)$ and $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$. Therefore, we have shown that $G((\mathrm{co}\mathbb{K}_0), 1) \subset \mathrm{co}G(\mathbb{K}_0, 1) + \mathrm{co}G(\mathbb{K}_0, 0)$. In the discussion above, we have implicitly assumed that $I_0 \neq \emptyset$; otherwise μ_j $(j \in I_0)$ cannot be consistently defined. If $I_0 = \emptyset$, we can just neglect μ_j and $\boldsymbol{Z}^j$ $(j \in I_0)$ and take $\boldsymbol{Z} = \boldsymbol{O}$. Then all the discussions above remain valid.

To show the converse inclusion, suppose that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$ for some $\boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}_0, 1)$ and $\boldsymbol{Z} \in \mathrm{co}G(\mathbb{K}_0, 0)$. Then we can represent $\boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}_0, 1)$ as

$$\boldsymbol{Y} \ = \ \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}^i, \ \sum_{i=1}^{p} \lambda_i = 1, \ \lambda_i > 0, \ \boldsymbol{Y}^i \in \mathbb{K}_0, \ \langle \boldsymbol{H}^0, \boldsymbol{Y}^i\rangle = 1 \ (i = 1, 2, \ldots, p),$$

and $\boldsymbol{Z} \in \mathrm{co}G(\mathbb{K}_0, 0)$ and

$$\boldsymbol{Z} \ = \ \sum_{j=1}^{q} \lambda_i \boldsymbol{Z}^j, \ \sum_{j=1}^{q} \lambda_j = 1, \ \lambda_j > 0, \ \boldsymbol{Z}^j \in \mathbb{K}_0, \ \langle \boldsymbol{H}^0, \boldsymbol{Z}^j\rangle = 0 \ (j = 1, 2, \ldots, q).$$

Since $\mathrm{co}\mathbb{K}_0$ is a convex cone, it follows from $\boldsymbol{Y} = \sum_{i=1}^{p} \lambda_i \boldsymbol{Y}^i \in \mathrm{co}\mathbb{K}_0$ and $\boldsymbol{Z} = \sum_{j=1}^{q} \lambda_j \boldsymbol{Z}^j \in \mathrm{co}\mathbb{K}_0$ that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z} \in \mathrm{co}\mathbb{K}_0$. We also see that

$$\langle \boldsymbol{H}^0, \boldsymbol{X}\rangle \ = \ \sum_{i=1}^{p} \lambda_i \langle \boldsymbol{H}^0, \boldsymbol{Y}^i\rangle + \sum_{j=1}^{q} \lambda_j \langle \boldsymbol{H}^0, \boldsymbol{Z}^j\rangle = \sum_{i=1}^{p} \lambda_i + 0 = 1.$$

Thus, we have shown that $\boldsymbol{X} \in G(\mathrm{co}\mathbb{K}_0, 1)$.

(ii) We see from (i) that

$$\begin{aligned}
\zeta(\mathrm{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) &= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Y} + \boldsymbol{Z}\rangle : \boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}_0, 1), \ \boldsymbol{Z} \in \mathrm{co}G(\mathbb{K}_0, 0) \right\} \\
&= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Y}\rangle : \boldsymbol{Y} \in \mathrm{co}G(\mathbb{K}_0, 1) \right\} + \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Z}\rangle : \boldsymbol{Z} \in \mathrm{co}G(\mathbb{K}_0, 0) \right\} \\
&= \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Y}\rangle : \boldsymbol{Y} \in G(\mathbb{K}_0, 1) \right\} + \inf \left\{ \langle \boldsymbol{Q}^0, \boldsymbol{Z}\rangle : \boldsymbol{Z} \in G(\mathbb{K}_0, 0) \right\} \\
&= \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) + \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0).
\end{aligned}$$

(iii) "if part": Assume that Condition II_0 holds. Then $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0) = 0$ follows from Lemma 3.5. Hence $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ by (ii). If $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = -\infty$, then $\zeta(\text{co } \mathbb{K}_0, \boldsymbol{Q}^0, 1) \leq \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = -\infty$.

"only if part": Assume that $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$. By Condition I_0, $G(\mathbb{K}_0, 1)$ is nonempty. Hence we have $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) < \infty$. If $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1) = -\infty$, then we are done. So suppose that $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 1)$ is finite. Then the assumption and (ii) implies that $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0) = 0$. Consequently, Condition II_0 holds. $\square$

Note that by (i) and (ii) of Lemma 3.1, we can replace $\text{co}G(\mathbb{K}_0, 0)$ and $\zeta(\mathbb{K}_0, \boldsymbol{Q}^0, 0)$ in Theorem 3.2 by $G(\text{co}\mathbb{K}_0, 0)$ and $\zeta(\text{co}\mathbb{K}_0, \boldsymbol{Q}^0, 0)$, respectively.

Next, we establish the following lemma which will play an essential role to extend Lemma 3.1 and Theorem 3.2 to a class of general COPs in the next section.

Lemma 3.4. *Let $\mathbb{K} \subset \mathbb{V}$ be a cone. Assume that $\mathbb{J}$ is a face of* $\text{co}\mathbb{K}$*. Then,*

(i) $\text{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J}$.

(ii) $(\mathbb{K} \cap \mathbb{J})^* = \mathbb{J}^*$.

Proof. (i) Since $\mathbb{J} = (\text{co}\mathbb{K}) \cap \mathbb{J}$ is a convex set containing $\mathbb{K} \cap \mathbb{J}$, we see $\text{co}(\mathbb{K} \cap \mathbb{J}) \subset \mathbb{J}$. To show the converse inclusion, let $\boldsymbol{X} \in \mathbb{J} = (\text{co}\mathbb{K}) \cap \mathbb{J}$. Then there exist $\boldsymbol{X}^i \in \mathbb{K} \subset \text{co}\mathbb{K}$ such that $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}^i$. Since $\mathbb{J}$ is a face of $\text{co}\mathbb{K}$, we see that $\boldsymbol{X}^i \in \mathbb{J}$ $(i = 1, \ldots, m)$. Therefore, $\boldsymbol{X}^i \in \mathbb{K} \cap \mathbb{J}$ $(i = 1, \ldots, m)$ and $\boldsymbol{X} = \sum_{i=1}^m \boldsymbol{X}^i \in \text{co}(\mathbb{K} \cap \mathbb{J})$.

(ii) Since $(\mathbb{K} \cap \mathbb{J})^* = (\text{co}(\mathbb{K} \cap \mathbb{J}))^*$ by Lemma 2.1, $(\mathbb{K} \cap \mathbb{J})^* = \mathbb{J}^*$ follows from (i). $\square$

3.2 A class of general conic optimization problems

For every cone $\mathbb{K} \subset \mathbb{V}$, every cone $\mathbb{J} \subset \mathbb{V}$ and every $\boldsymbol{Q}^0 \in \mathbb{V}$, we consider the class of general COPs of the following form:

$$\begin{aligned} \text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)\text{:} \quad \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0, 1) &= \inf \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{X} \rangle : \boldsymbol{X} \in G(\mathbb{K} \cap \mathbb{J}, 1) \right\} \\ &= \inf \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}, \langle \boldsymbol{H}^0, \, \boldsymbol{X} \rangle = 1 \right\}. \end{aligned}$$

Obviously, we can handle $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ as a special case of $\text{COP}(\mathbb{K}_0, \boldsymbol{Q}^0)$ by taking $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$. In particular, we can apply Lemma 3.1 and Theorem 3.2 if we assume Conditions I_0 and II_0 for $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$. We further impose the condition that $\mathbb{J}$ is a face of $\text{co}\mathbb{K}$, which would provide various interesting structures in $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and a bridge between Theorem 3.2 and many existing results on the convexification of nonconvex quadratic and polynomial optimization problems. By (ii) of Lemma 3.4, we know that $\mathbb{K}_0^* = (\mathbb{K} \cap \mathbb{J})^* = \mathbb{J}^*$ under the assumption. Thus we can replace Conditions I_0 and II_0 by the following Conditions 0_{J}, I_{J} and II_{J} for $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$.

Condition $\mathbf{0}_{\text{J}}$: $\mathbb{J}$ is a face of $\text{co}\mathbb{K}$.

Condition $\mathbf{I}_{\text{J}}$: $\text{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ is feasible, *i.e.*, $G(\mathbb{K} \cap \mathbb{J}, 1) \neq \emptyset$, and $\boldsymbol{O} \neq \boldsymbol{H}^0 \in \mathbb{J}^*$.

Condition $\mathbf{II}_{\text{J}}$: $\inf \left\{ \langle \boldsymbol{Q}^0, \, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}, \ \langle \boldsymbol{H}^0, \, \boldsymbol{X} \rangle = 0 \right\} \geq 0$, *i.e.*, $\zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0, 0) \geq 0$.

Note that Condition 0_{J} is newly added while Conditions I_{J} and II_{J} are equivalent to Conditions I_0 and II_0 with $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$ under Condition 0_{J}, respectively.

Let $\mathbb{J}$ be a face of $\mathrm{co}\mathbb{K}$ and $\mathbb{K}_0 = \mathbb{K} \cap \mathbb{J}$. Then, we know by (i) of Lemma 3.4 that $\mathrm{co}\mathbb{K}_0 = \mathrm{co}(\mathbb{K} \cap \mathbb{J}) = \mathbb{J}$. Replacing $\mathbb{K}_0$ by $\mathbb{K} \cap \mathbb{J}$ and $\mathrm{co}\mathbb{K}_0$ by $\mathbb{J}$ in Lemma 3.1 and in Theorem 3.2, we obtain the following results in Lemma 3.5 and Theorem 3.6.

Lemma 3.5. *Let $\mathbb{K} \subset \mathbb{V}$ be a cone. Assume that Conditions 0_{J} and I_{J} hold. Then,*

(i) $\mathrm{co}G(\mathbb{K} \cap \mathbb{J}, 0) = G(\mathbb{J}, 0)$.

(ii) For every $\boldsymbol{P} \in \mathbb{V}$, $\zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{P}, 0) = \zeta(\mathbb{J}, \boldsymbol{P}, 0)$.

(iii) For every $\boldsymbol{P} \in \mathbb{V}$, $\zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{P}, 0) = \begin{cases} 0 & \textit{if Condition } \mathit{II}_{\mathrm{J}} \textit{ holds}, \\ -\infty & \textit{otherwise.} \end{cases}$

Theorem 3.6. *Let $\mathbb{K} \subset \mathbb{V}$ be a cone and $\boldsymbol{Q}^0 \in \mathbb{V}$. Assume that Conditions 0_{J} and I_{J} hold. Then,*

(i) $G(\mathbb{J}, 1) = \mathrm{co}G(\mathbb{K} \cap \mathbb{J}, 1) + \mathrm{co}G(\mathbb{K} \cap \mathbb{J}, 0)$.

(ii) $\zeta(\mathbb{J}, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0, 1) + \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0, 0)$.

(iii) $\zeta(\mathbb{J}, \boldsymbol{Q}^0, 1) = \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0, 1)$ *iff*

$$\textit{Condition } \mathit{II}_{\mathrm{J}} \textit{ or } \zeta(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0, 1) = -\infty \textit{ holds.} \tag{10}$$

The assertions of Lemma 3.5 and Theorem 3.6 are similar to but more general than those of Lemma 3.1 and Theorem 3.1 of [3], respectively. The essential difference is that our results here cover the case where $\mathbb{J}$ can be a non-exposed face while those in [3] are restricted to the case where $\mathbb{J}$ is an exposed face of $\mathrm{co}\mathbb{K}$ which is represented explicitly as $\mathbb{J} = \{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \ (p = 1, \ldots, m)\}$ for some $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \ldots, m)$.

Suppose that $\mathbb{J}$ is a face of $\mathrm{co}\mathbb{K}$ and that its tangent space $\mathcal{T}(\mathbb{J})$ is represented as $\mathcal{T}(\mathbb{J}) = \{\boldsymbol{X} \in \mathbb{V} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \ (p = 1, \ldots, m)\}$ for some $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$. Then $\mathbb{J} = \mathrm{co}\mathbb{K} \cap \mathcal{T}(\mathbb{J})$ and the cone $\mathbb{J}$ is represented as in (3). Therefore, $\mathrm{COP}(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$ is equivalent to COP (2) introduced in Section 1. We note, however, that $\boldsymbol{Q}^p \in \mathbb{K}^*$ $(p = 1, \ldots, m)$ may not be satisfied.

Conversely, suppose that a COP of the form (2) is given. It is interesting to characterize a collection of $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ which induces a face $\mathbb{J}$ of $\mathrm{co}\mathbb{K}$. Such a characterization is necessary to construct a class of COPs of the form (2) that can be reformulated as convex COPs. One sufficient condition (which was assumed in [5, 3, 19]) for $\mathbb{J}$ defined by (3) to be a face of $\mathrm{co}\mathbb{K}$ is that all $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ are copositive on $\mathbb{K}$. However, this sufficient condition can sometimes be restrictive. For example, we can replace $\boldsymbol{Q}^m$ by $-\sum_{p=1}^m \boldsymbol{Q}^p$ to generate the same $\mathbb{J}$ but $-\sum_{p=1}^m \boldsymbol{Q}^p$ is no longer copositive on $\mathbb{K}$. We also see that this sufficient condition ensures that $\mathbb{J}$ defined by (3) is an exposed face of $\mathrm{co}\mathbb{K}$. In fact, in this case, $\mathbb{J}$ coincides with $\left\{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \sum_{p=1}^m \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0\right\}$. If $\mathbb{J}$ is a non-exposed face of $\mathrm{co}\mathbb{K}$, $\mathbb{J}$ cannot be represented in terms of any collection of copositive $\boldsymbol{Q}^p$ on $\mathbb{K}$ $(p = 1, \ldots, m)$ as in (3). We will investigate such cases in Section 4.

4 Copositivity conditions

Throughout this section, we fix a linear space $\mathbb{V}$, a cone $\mathbb{K} \subset \mathbb{V}$ and $\boldsymbol{H}^0 \in \mathbb{V}$. In Section 3.2, we have shown that if $\mathbb{J}$ is a face of co$\mathbb{K}$, we can always represent $\mathbb{J}$ as in (3) for some $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$. In Section 4.1, we strengthen this equivalence relation by introducing a *hierarchy of copositivity condition* and show how we can choose such $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ to satisfy the condition recursively. The hierarchy of copositivity condition was originally proposed in Arima, Kim and Kojima [1] as a condition for characterizing a class of QOPs that are equivalent to their CPP reformulations. Here, we extend the condition to a more general class of COP$(\mathbb{K} \cap \mathbb{J}, \boldsymbol{Q}^0)$, which includes their class of QOPs. In Section 4.2, we present some characterizations of the copositivity of $\boldsymbol{P} \in \mathbb{V}$ on a face $\mathbb{J}$ of co$\mathbb{K}$. They are useful to construct a face $\mathbb{J}$ of co$\mathbb{K}$ in terms of $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 1, \ldots, m)$ as in (3).

4.1 The hierarchy of copositivity condition

Recall that $\mathbb{J}$ is an exposed face of co$\mathbb{K}$ iff $\mathbb{J} = \left\{ \boldsymbol{X} \in \text{co}\mathbb{K} : \langle \boldsymbol{Q}^1, \boldsymbol{X} \rangle = 0 \right\}$ for some copositive $\boldsymbol{Q}^1 \in \mathbb{V}$ on co$\mathbb{K}$, *i.e.*, $\boldsymbol{Q}^1 \in (\text{co}\mathbb{K})^*$. The two lemmas below generalize this fact, assuming implicitly that $\mathbb{J}$ can be a non-exposed face of co$\mathbb{K}$. (As we have mentioned in Section 1, some faces of the CPP cone $\mathbb{CPP}^{1+n}$ are non-exposed if $n \geq 5$ [29].)

Lemma 4.1. *Let $\mathbb{J}$ be a proper face of co$\mathbb{K}$. Let $\mathbb{J}_0 =$ co$\mathbb{K}$. Then there exist sequences of faces $\mathbb{J}_1, \ldots, \mathbb{J}_m$ of co$\mathbb{K}$ and $\boldsymbol{Q}^1, \ldots, \boldsymbol{Q}^m \in \mathbb{V}$ for some positive integer m such that*

$$\left. \begin{array}{l} \boldsymbol{O} \neq \boldsymbol{Q}^p \in \mathbb{J}^*_{p-1} \cap \mathcal{T}(\mathbb{J}_{p-1}), \\ \mathbb{J}_p = \left\{ \boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \right\}, \text{dim}\mathbb{J}_{p-1} > \text{dim}\mathbb{J}_p \textit{ and } \mathbb{J}_m = \mathbb{J} \end{array} \right\} \tag{11}$$

$(p = 1, \ldots, m)$.

Proof. Let $\overline{\boldsymbol{X}}$ be a relative interior point of $\mathbb{J}$ with respect to the tangent space $\mathcal{T}(\mathbb{J})$ of $\mathbb{J}$. Since $\overline{\boldsymbol{X}}$ is a boundary point of the cone $\mathbb{J}_0 =$ co$\mathbb{K}$ with respect to the tangent space $\mathcal{T}(\mathbb{J}_0)$, we can take a supporting hyperplane of $\mathbb{J}_0$ at $\overline{\boldsymbol{X}}$ in the tangent space $\mathcal{T}(\mathbb{J}_0)$, say, $\left\{ \boldsymbol{X} \in \mathcal{T}(\mathbb{J}_0) : \langle \boldsymbol{Q}^1, \boldsymbol{X} \rangle = 0 \right\}$ for some nonzero $\boldsymbol{Q}^1 \in \mathbb{J}^*_0 \cap \mathcal{T}(\mathbb{J}_0)$. Let $\mathbb{J}_1 = \left\{ \boldsymbol{X} \in \mathbb{J}_0 : \langle \boldsymbol{Q}^1, \boldsymbol{X} \rangle = 0 \right\}$, which forms a face of $\mathbb{J}_0 =$ co$\mathbb{K}$ by (ii) of Lemma 2.1. By construction, $\mathbb{J} \subset \mathbb{J}_1 \subset \mathbb{J}_0$ and $\text{dim}\mathbb{J} \leq \text{dim}\mathbb{J}_1 < \text{dim}\mathbb{J}_0$. If $\mathbb{J} = \mathbb{J}_1$, $\mathcal{T}(\mathbb{J}) = \mathcal{T}(\mathbb{J}_1)$ or $\overline{\boldsymbol{X}}$ lies in the relative interior of $\mathbb{J}_1$ with respect to $\mathcal{T}(\mathbb{J}_1)$, we are done. In general, suppose that $\overline{\boldsymbol{X}}$ is a relative boundary point of a face $\mathbb{J}_{p-1}$ with respect to $\mathcal{T}(\mathbb{J}_{p-1})$ $(1 \leq p)$, we can take a supporting hyperplane of $\mathbb{J}_{p-1}$ at $\overline{\boldsymbol{X}}$ in the tangent space $\mathcal{T}(\mathbb{J}_{p-1})$, say $\left\{ \boldsymbol{X} \in \mathcal{T}(\mathbb{J}_{p-1}) : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \right\}$ for some nonzero $\boldsymbol{Q}^p \in \mathbb{J}^*_{p-1} \cap \mathcal{T}(\mathbb{J}_{p-1})$. Let $\mathbb{J}_p = \left\{ \boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle = 0 \right\}$. Since $\text{dim}\mathbb{J}_{p-1} > \text{dim}\mathbb{J}_p$, this process terminates in a finite number of steps to obtain a sequence of faces $\mathbb{J}_1, \ldots, \mathbb{J}_m$ of co$\mathbb{K}$ and a sequence $\boldsymbol{Q}^1, \ldots, \boldsymbol{Q}^m \in \mathbb{V}$ satisfying (11). $\square$

Note that Lemma 4.1 shows that any proper face $\mathbb{J}$ of co$\mathbb{K}$ can be represented in terms of *a hierarchy of copositivity condition*:

$$\begin{aligned}
\mathbb{J}_0 &= \text{co}\mathbb{K}, \quad (12)\\
\mathbb{J}_p &= \left\{ \boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^{pj},\, \boldsymbol{X} \rangle = 0 \ (j = 1, \ldots, q_p) \right\} \\
&\quad \text{for some copositive } \boldsymbol{Q}^{pj} \in \mathbb{V} \ (j = 1, \ldots, q_p) \text{ on } \mathbb{J}_{p-1} \ (p = 1, \ldots, m), \quad (13)\\
\mathbb{J} &= \mathbb{J}_m = \left\{ \boldsymbol{X} \in \text{co}\mathbb{K} : \langle \boldsymbol{Q}^{pj},\, \boldsymbol{X} \rangle = 0 \ (j = 1, \ldots, q_p, p = 1, \ldots, m) \right\} \quad (14)
\end{aligned}$$

for some positive integers q_p $(p = 1, \ldots, m)$ and m.

Conversely, we can construct any face $\mathbb{J}$ of co$\mathbb{K}$ by (12), (13) and (14) as we shall present next. Since all $\boldsymbol{Q}^{pj} \in \mathbb{V}$ $(j = 1, \ldots, q_p)$ are copositive on $\mathbb{J}_{p-1}$ in (13), we can replace (13) by

$$\begin{aligned}
\mathbb{J}_p &= \left\{ \boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^{p},\, \boldsymbol{X} \rangle = 0 \right\} \\
&\quad \text{for some copositive } \boldsymbol{Q}^{p} \in \mathbb{V} \text{ on } \mathbb{J}_{p-1} \ (p = 1, \ldots, m) \quad (15)
\end{aligned}$$

as in Lemma 4.1 by letting $\boldsymbol{Q}^p = \sum_{j=1}^{q_p} \boldsymbol{Q}^{qj}$. We also see that if $0 \leq k < p \leq m$ and $\boldsymbol{P} \in \mathbb{V}$ is copositive on $\mathbb{J}_k$, then it is copositive on $\mathbb{J}_p$ since $(\mathbb{J}_k)^* \supset (\mathbb{J}_p)^*$. This implies that replacing (13) by (15) is not restrictive at all. Furthermore, if $\mathbb{J}_{p-1}$ is a face of co$\mathbb{K}$, then $\mathbb{J}_{p-1} = \text{co}(\mathbb{K} \cap \mathbb{J}_{p-1})$ by (i) of Lemma 3.4. Hence, "copositive on $\mathbb{J}_{p-1}$" can be replaced by "copositive on $\mathbb{K} \cap \mathbb{J}_{p-1}$" in (13) and (15).

Lemma 4.2. *Let $\mathbb{K} \subset \mathbb{V}$ be a cone. Let $\boldsymbol{Q}^p \in \mathbb{V}$ $(p = 0, \ldots, m)$ be given, and construct a sequence of $\mathbb{J}_p \subset \mathbb{V}$ $(p = 0, \ldots, m)$ by*

$$\mathbb{J}_0 = \textit{co}\mathbb{K} \textit{ and } \mathbb{J}_p = \left\{ \boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^{p},\, \boldsymbol{X} \rangle = 0 \right\} \ (p = 1, \ldots, m) \quad (16)$$

Assume that $\boldsymbol{Q}^p \in \mathbb{V}$ is copositive on $\mathbb{K} \cap \mathbb{J}_{p-1}$ $(p = 1, \ldots, m)$. Then each $\mathbb{J}_p$ is a face of $\mathbb{J}_{p-1}$ and a face of co$\mathbb{K}$ $(p = 1, \ldots, m)$.

Proof. The assertion follows from (ii) and (iii) of Lemma 2.1. □

It should be noted that $\boldsymbol{Q}^1$ need to be chosen from the cone $\mathbb{K}^*$, but $\boldsymbol{Q}^p$ from a possibly wider cone $\mathbb{J}^*_{p-1}$ than $\mathbb{J}^*_{p-2}$ $(p = 2, \ldots, m)$.

4.2 Characterization of copositivity

Let $\mathbb{J}_0 = \text{co}\mathbb{K}$. We assume that $k = 0$ or a face $\mathbb{J}_k$ of $\mathbb{J}_{k-1}$ has already been constructed through (16) for some $k = 1, \ldots, p-1$. Now, we focus on the choice of a copositive $\boldsymbol{Q}^p \in \mathbb{V}$ on $\mathbb{K} \cap \mathbb{J}_{p-1}$, *i.e.*, $\boldsymbol{Q}^p \in (\mathbb{K} \cap \mathbb{J}_{p-1})^*$ so that the cone $\mathbb{J}_p = \left\{ \boldsymbol{X} \in \mathbb{J}_{p-1} : \langle \boldsymbol{Q}^{p},\, \boldsymbol{X} \rangle = 0 \right\}$ can become a face of $\mathbb{J}_{p-1}$. By definition, $\boldsymbol{Q}^p$ is copositive on $\mathbb{K} \cap \mathbb{J}_{p-1}$ iff

$$\eta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) \equiv \inf \left\{ \langle \boldsymbol{Q}^{p},\, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{K} \cap \mathbb{J}_{p-1} \right\} \geq 0. \quad (17)$$

Lemma 4.3. *Let $\boldsymbol{H}^0 \in \mathbb{K}^*$ and $\mathbb{J}_{p-1}$ be a face of co$\mathbb{K}$. Assume that $\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) \geq 0$. Then, (17) is equivalent to either of the following two conditions:*

$$\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, \rho) \geq 0 \textit{ for every } \rho \geq 0, \quad (18)$$

$$\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 1) \geq 0. \quad (19)$$

Proof. Since $\mathbb{K}\cap\mathbb{J}_{p-1} \supset G(\mathbb{K}\cap\mathbb{J}_{p-1},\rho)$ for every $\rho \geq 0$, we see that (17) $\Rightarrow$ (18) $\Rightarrow$ (19). Thus, it suffices to show that (19) $\Rightarrow$ (18) $\Rightarrow$ (17).

(19) $\Rightarrow$ (18): Assume that (19) holds. Let $\boldsymbol{X} \in \mathbb{K}\cap\mathbb{J}_{p-1}$ and $\langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle = \rho$. First, we consider the case $\rho > 0$. Then, $\langle \boldsymbol{H}^0,\, \boldsymbol{X}/\rho\rangle = 1$ and $\boldsymbol{X}/\rho \in \mathbb{K}\cap\mathbb{J}_{p-1}$. As a result, $\langle \boldsymbol{Q}^p,\, \boldsymbol{X}/\rho\rangle \geq 0$, which implies that $\langle \boldsymbol{Q}^p,\, \boldsymbol{X}\rangle \geq 0$. Therefore $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, \rho) \geq 0$. The second case where $\rho = 0$ simply follows from the assumption that $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) \geq 0$.

(18) $\Rightarrow$ (17): Assume that (18) holds. Take $\boldsymbol{X} \in \mathbb{K}\cap\mathbb{J}_{p-1}$ arbitrarily. It follows from $\boldsymbol{X} \in \mathbb{J}_{p-1}$ and $\boldsymbol{H}^0 \in \mathbb{K}^* \subset \mathbb{J}^*_{p-1}$ that $\rho = \langle \boldsymbol{H}^0,\, \boldsymbol{X}\rangle \geq 0$. Hence $\boldsymbol{X} \in G(\mathbb{K}\cap\mathbb{J}_{p-1},\rho)$ with $\rho \geq 0$. Thus $\langle \boldsymbol{Q}^p,\, \boldsymbol{X}\rangle \geq 0$ follows from (18). $\square$

Remark 4.4. Suppose that (19) holds. If $G(\mathbb{K}\cap\mathbb{J}_p, 1) \neq \emptyset$, then $\langle \boldsymbol{Q}^p,\, \boldsymbol{X}\rangle = 0$ for some $\boldsymbol{X} \in G(\mathbb{K}\cap\mathbb{J}_{p-1}, 1)$. Thus, $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, 1) = 0$ and $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, \rho) = 0$ for every $\rho > 0$. This implies that $\lim_{\rho\to 0+} \zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, \rho) = 0$. By (iii) of Lemma 3.5, we also know that either $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) = 0$ or $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) = -\infty$. Thus the assumption made in Lemma 4.3 is to ensure that the latte case where $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) = -\infty < \lim_{\rho\to 0+} \zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, \rho) = 0$ (a discontinuity of $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, \rho)$ at $\rho = 0$) cannot occur. Note that if $G(\mathbb{K}\cap\mathbb{J}_{p-1}, 0) = \{\boldsymbol{0}\}$, then clearly $\zeta(\mathbb{K}\cap\mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) = 0$, and the assumption of the lemma holds.

Remark 4.5. We consider the case where $\mathbb{J}_{p-1}$ is an exposed face of $\mathrm{co}\mathbb{K}$, where there is a nonzero $\boldsymbol{H}^1 \in \mathbb{K}^*$ such that $\mathbb{J}_{p-1} = \mathrm{co}\mathbb{K}\cap\mathbb{L}$ with $\mathbb{L} = \left\{\boldsymbol{X} \in \mathbb{V} : \langle \boldsymbol{H}^1,\, \boldsymbol{X}\rangle = 0\right\}$. Then $\mathbb{J}^*_{p-1} = \mathrm{cl}(\mathbb{K}^* + \mathbb{L}^\perp)$. Now assume that $\boldsymbol{Q}^p \in \mathbb{V}$ is copositive on $\mathbb{K}\cap\mathbb{J}_{p-1}$ or $\boldsymbol{Q}^p \in (\mathbb{K}\cap\mathbb{J}_{p-1})^* = \mathbb{J}^*_{p-1}$. If $\mathbb{K}^* + \mathbb{L}^\perp$ is closed, then $\mathbb{J}^*_{p-1} = \mathbb{K}^* + \mathbb{L}^\perp$. Hence there exist $\widehat{\boldsymbol{Y}} \in \mathbb{K}^*$ and $\hat{y}_1 \in \mathbb{R}$ such that $\boldsymbol{Q}^p = \widehat{\boldsymbol{Y}} + \boldsymbol{H}^1\hat{y}_1$. It follows that

$$
\begin{aligned}
\mathbb{J}_p &= \left\{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{H}^1,\, \boldsymbol{X}\rangle = 0,\ \langle \boldsymbol{Q}^p,\, \boldsymbol{X}\rangle = 0\right\} \\
&= \left\{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{H}^1,\, \boldsymbol{X}\rangle = 0,\ \langle \widehat{\boldsymbol{Y}} + \boldsymbol{H}^1\hat{y}_1,\, \boldsymbol{X}\rangle = 0\right\} \\
&= \left\{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{H}^1,\, \boldsymbol{X}\rangle = 0,\ \langle \widehat{\boldsymbol{Y}},\, \boldsymbol{X}\rangle = 0\right\} \\
&= \left\{\boldsymbol{X} \in \mathrm{co}\mathbb{K} : \langle \boldsymbol{H}^1 + \widehat{\boldsymbol{Y}},\, \boldsymbol{X}\rangle = 0\right\}. \ (\text{since } \boldsymbol{H}^1,\ \widehat{\boldsymbol{Y}} \in \mathbb{K}^*)
\end{aligned}
$$

This implies that $\mathbb{J}_p$ is also an exposed face of $\mathrm{co}\mathbb{K}$.

5 Convex COP reformulation of polynomial optimization problems

We extend the CPP reformulation of QOPs studied in many papers such as [10, 8, 11] (see also Sections 2.2 and 6.1.) to POPs. The results presented in this section are closely related to those in Section 3 of [4], but our class of POPs of the form (1) that can be reformulated as convex COPs does cover POPs in nonnegative variables with polynomial equality constraints satisfying the hierarchy of copositivity conditions, which is more general than the copositivity condition assumed in Section 3 of [4].

To apply the results described in Sections 3 and 4 to a convex conic reformulation of POP (1), we first reduce POP (1) to $\mathrm{COP}(\boldsymbol{\Gamma}^{\mathcal{A}}\cap\mathbb{J}, \boldsymbol{Q}^0)$. Here $\boldsymbol{\Gamma}^{\mathcal{A}}$ is a nonconvex cone in a linear space $\mathbb{S}^{\mathcal{A}}$ of symmetric matrices whose dimension depends on the maximum degree of the monomials involved in $f_i(\boldsymbol{w})$ $(i = 0, \ldots, m)$ of POP (1), and $\mathcal{A}$ stands for a set of monomials. The convex hull of $\boldsymbol{\Gamma}^{\mathcal{A}}$, denoted as $\mathbb{CPP}^{\mathcal{A}}$, corresponds to an extension of the CPP cone $\mathbb{CPP}^{1+n}$. The polynomial function $f_p(\boldsymbol{w})$ is converted into $\langle \boldsymbol{Q}^p, \boldsymbol{X} \rangle$ in $\boldsymbol{X} \in \boldsymbol{\Gamma}^{\mathcal{A}}$ for some $\boldsymbol{Q}^p \in \mathbb{S}^{\mathcal{A}}$ with the additional constraint $\langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1$ through its homogenization $\bar{f}_p(\boldsymbol{x})$ $(p = 0, \ldots, m)$, and then the face $\mathbb{J}$ of $\mathbb{CPP}^{\mathcal{A}}$ is defined as in (3).

We explain how a polynomial function in $\boldsymbol{w} \in \mathbb{R}^n$ is homogenized in Section 5.1, and define an extended completely positive cone $\mathbb{CPP}^{\mathcal{A}}$ in Section 5.2. The conversion of POP (1) into COP($\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0$) is presented in Section 5.3, and the convex reformulation of COP($\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0$) into COP($\mathbb{J}, \boldsymbol{Q}^0$) is discussed in Section 5.4.

5.1 Homogenizing polynomial functions

Let τ be a positive integer. We call that a real valued polynomial function $\bar{f}(\boldsymbol{x})$ in $\boldsymbol{x} \in \mathbb{R}^{1+n}$ is *homogeneous* with degree $\tau \in \mathbb{Z}_+$ (or degree τ homogeneous) if $\bar{f}(\lambda \boldsymbol{x}) = \lambda^\tau \bar{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}$ and $\lambda \geq 0$. For the consistency of the discussions throughout Section 5, a homogeneous polynomial function is defined in $\mathbb{R}^{1+n}$ but not $\mathbb{R}^n$, where the first coordinate of $\mathbb{R}^{1+n}$ is indexed by 0; we write $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ or $\boldsymbol{x} = (x_0, \boldsymbol{w})$ with $\boldsymbol{w} \in \mathbb{R}^n$.

For each $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, let $\boldsymbol{w}^{\boldsymbol{\alpha}}$ denote the monomial $\prod_{i=1}^n w_i^{\alpha_i}$ with degree $\tau_0 = |\boldsymbol{\alpha}| \equiv \sum_{i=1}^n \alpha_i$. Let τ be a nonnegative integer no less than τ_0. By introducing an additional variable $x_0 \in \mathbb{R}$, which will be fixed to 1 later, we can convert the previous monomial to the monomial $x_0^{\tau - \tau_0} \boldsymbol{w}^{\boldsymbol{\alpha}}$ in $(x_0, w_1, \ldots, w_n) \in \mathbb{R}^{1+n}$ with degree τ. Using this technique, we can convert any polynomial function $f(\boldsymbol{w})$ in $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ with degree τ_0 to a homogeneous polynomial function $\bar{f}(x_0, \boldsymbol{w})$ in $(x_0, w_1, \ldots, w_n) \in \mathbb{R}^{1+n}$ with degree $\tau \geq \tau_0$ such that $\bar{f}(1, \boldsymbol{w}) = f(\boldsymbol{w})$ for every $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$.

5.2 An extension of the completely positive cone

We begin by introducing some additional notation and symbols. For each positive integer ω, we define $\mathcal{A}_\omega = \left\{ \boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{1+n}_+ : |\boldsymbol{\alpha}| = \omega \right\}$. For each nonempty subset $\mathcal{A}$ of $\mathcal{A}_\omega$, let $\mathbb{R}^{\mathcal{A}}$ be the $|\mathcal{A}|$-dimensional Euclidean space whose coordinates are indexed by $\boldsymbol{\alpha} \in \mathcal{A}$, where $|\mathcal{A}|$ stands for the cardinality of $\mathcal{A}$, *i.e.*, the number of elements in $\mathcal{A}$. We use $\mathbb{S}^{\mathcal{A}} \subset \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ to denote the space of $|\mathcal{A}| \times |\mathcal{A}|$ symmetric matrices whose elements are indexed by $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{A} \times \mathcal{A}$. Let $\mathbb{S}^{\mathcal{A}}_+$ denote the cone of positive semidefinite matrices in $\mathbb{S}^{\mathcal{A}}$, and $\mathbb{N}^{\mathcal{A}}$ the cone of nonnegative matrices in $\mathbb{S}^{\mathcal{A}}$.

Let ω be a positive integer and $\emptyset \neq \mathcal{A} \subset \mathcal{A}_\omega$. We define

$$\boldsymbol{\Gamma}^{\mathcal{A}} = \left\{ \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in \mathbb{S}^{\mathcal{A}} : \boldsymbol{x} \in \mathbb{R}^{1+n}_+ \right\} \text{ and } \mathbb{CPP}^{\mathcal{A}} = \mathrm{co}\boldsymbol{\Gamma}^{\mathcal{A}}.$$

Here $\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})$ denotes the $|\mathcal{A}|$-dimensional column vector of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathcal{A})$. We note that every element $[\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T]_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\beta}}$ is a degree 2ω monomial in $\boldsymbol{x} \in \mathbb{R}^{1+n}$. It follows that $\boldsymbol{\Gamma}^{\mathcal{A}}$ forms a cone in $\mathbb{S}^{\mathcal{A}}$. The coordinate indices $(\boldsymbol{\alpha} \in \mathcal{A})$ are ordered so that $\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}) \in \mathbb{R}^{\mathcal{A}}$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}$. We call $\mathbb{CPP}^{\mathcal{A}}$ an *extended completely positive cone*, and the dual of $\mathbb{CPP}^{\mathcal{A}}$, $\mathbb{COP}^{\mathcal{A}} = (\mathbb{CPP}^{\mathcal{A}})^* = (\boldsymbol{\Gamma}^{\mathcal{A}})^*$ an *extended copositive cone.*

By definition, we know that $\mathbf{\Gamma}^{\mathcal{A}} \subset \mathbb{S}^{\mathcal{A}} \cap \mathbb{N}^{\mathcal{A}}$. We also observe that

$$X_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\gamma}+\boldsymbol{\delta}} = X_{\boldsymbol{\gamma}\boldsymbol{\delta}} \text{ if } \boldsymbol{\alpha}+\boldsymbol{\beta} = \boldsymbol{\gamma}+\boldsymbol{\delta}$$

for every $\boldsymbol{X} = \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in \mathbf{\Gamma}^{\mathcal{A}}$. This implies that the cone $\mathbf{\Gamma}^{\mathcal{A}}$ and its convex hull $\mathbb{CPP}^{\mathcal{A}}$ is contained in the linear subspace $\mathbb{L}^{\mathcal{A}}$ of $\mathbb{S}^{\mathcal{A}}$ defined by

$$\mathbb{L}^{\mathcal{A}} \;=\; \left\{ \boldsymbol{X} \in \mathbb{S}^{\mathcal{A}} : X_{\boldsymbol{\alpha}\boldsymbol{\beta}} = X_{\boldsymbol{\gamma}\boldsymbol{\delta}} \text{ if } \boldsymbol{\alpha}+\boldsymbol{\beta} = \boldsymbol{\gamma}+\boldsymbol{\delta} \right\}.$$

Therefore,

$$\mathbf{\Gamma}^{\mathcal{A}} \subset \mathbb{CPP}^{\mathcal{A}} \subset \mathbb{S}^{\mathcal{A}}_+ \cap \mathbb{N}^{\mathcal{A}} \cap \mathbb{L}^{\mathcal{A}} \subset \mathbb{S}^{\mathcal{A}}_+ \subset \mathbb{S}^{\mathcal{A}}_+ + \mathbb{N}^{\mathcal{A}} + \left(\mathbb{L}^{\mathcal{A}}\right)^{\perp} \subset \mathbb{COP}^{\mathcal{A}} = (\mathbf{\Gamma}^{\mathcal{A}})^*. \tag{20}$$

Let $\bar{f}(\boldsymbol{x})$ be a degree 2ω homogeneous polynomial function. Then, we can write $\bar{f}(\boldsymbol{x}) = \sum_{\boldsymbol{\gamma}\in\mathcal{B}} c_{\boldsymbol{\gamma}} \boldsymbol{x}^{\boldsymbol{\gamma}}$ for some nonzero $c_{\boldsymbol{\gamma}} \in \mathbb{R}$ ($\boldsymbol{\gamma} \in \mathcal{B}$) and some $\mathcal{B} \subset \mathcal{A}_{2\omega}$. Since $\mathcal{A}_\omega + \mathcal{A}_\omega \equiv \{\boldsymbol{\alpha}+\boldsymbol{\beta} : \boldsymbol{\alpha} \in \mathcal{A}_\omega,\ \boldsymbol{\beta} \in \mathcal{A}_\omega\} = \mathcal{A}_{2\omega} \supset \mathcal{B}$ and the matrix $\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in \mathbb{S}^{\mathcal{A}}$ involves all monomials in $\mathcal{A}+\mathcal{A}$ for every $\mathcal{A} \subset \mathcal{A}_\omega$, we can choose an $\mathcal{A} \subset \mathcal{A}_\omega$ such that $\mathcal{B} \subset \mathcal{A}+\mathcal{A}$ (see [20] for such a choice $\mathcal{A}$ from $\mathcal{A}_\omega$), and a matrix $\boldsymbol{P} \in \mathbb{S}^{\mathcal{A}}$ such that $\bar{f}(\boldsymbol{x}) = \langle \boldsymbol{P},\, \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \rangle$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}$. (Note that such a $\boldsymbol{P} \in \mathbb{S}^{\mathcal{A}}$ is not unique.) In our subsequent discussion, we impose an additional condition that $\mathcal{A}$ contains $\boldsymbol{\alpha}^\omega \equiv (\omega, 0, \ldots, 0) \in \mathbb{R}^n$, and assume that the first coordinate of $\mathbb{R}^{\mathcal{A}}$ is $\boldsymbol{\alpha}^\omega$, the upper-leftmost element of each $\boldsymbol{X} \in \mathbb{S}^{\mathcal{A}} \subset \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ is $X_{\boldsymbol{\alpha}^\omega\boldsymbol{\alpha}^\omega}$ and that the first element of $\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}) \in \mathbb{R}^{\mathcal{A}}$ is $\boldsymbol{x}^{\boldsymbol{\alpha}^\omega} = x_0^\omega$.

As a consequence of the representation of $\bar{f}(\boldsymbol{x}) = \langle \boldsymbol{P},\, \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})^T \rangle$, $\bar{f}(\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}_+$ iff $\langle \boldsymbol{P},\, \boldsymbol{X} \rangle \geq 0$ for every $\boldsymbol{X} \in \mathbf{\Gamma}^{\mathcal{A}}$ or equivalently $\boldsymbol{P} \in (\mathbf{\Gamma}^{\mathcal{A}})^* = \mathbb{COP}^{\mathcal{A}}$.

5.3 Conversion of POP (1) to COP($\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0$)

Let $\tau_{\min} = \max\{\deg f_i(\boldsymbol{w}) : i = 0, \ldots, m\}$ and ω be a positive integer such that $2\omega \geq \tau_{\min}$. By applying the homogenization technique with degree 2ω described in the previous section to the polynomial function $f_i(\boldsymbol{x})$ ($i = 0, \ldots, m$), we can convert POP (1) to

$$\zeta^* = \inf \left\{ \bar{f}_0(\boldsymbol{x}) : \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n},\ \bar{f}_i(\boldsymbol{x}) = 0 \ (i = 1, \ldots, m),\ x_0 = 1 \right\}. \tag{21}$$

Here $\bar{f}_i(\boldsymbol{x})$ denotes a degree 2ω homogeneous polynomial function in $\boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}$ such that $\bar{f}_i(1, \boldsymbol{w}) = f_i(\boldsymbol{w})$ for every $\boldsymbol{w} \in \mathbb{R}^n$ ($i = 0, \ldots, m$).

As discussed in the previous subsection, we choose an $\mathcal{A} \subset \mathcal{A}_\omega$ such that $\boldsymbol{\alpha}^\omega \equiv (\omega, 0, \ldots, 0) \in \mathcal{A}$ and the set of monomials $\{\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \boldsymbol{\alpha} \in \mathcal{A},\ \boldsymbol{\beta} \in \mathcal{A}\}$ covers all monomials involved in $\bar{f}_i(\boldsymbol{x})$ ($i = 0, \ldots, m$), and choose $\boldsymbol{Q}^i \in \mathbb{S}^{\mathcal{A}}$ ($i = 0, \ldots, m$) to satisfy

$$\bar{f}_i(\boldsymbol{x}) = \langle \boldsymbol{Q}^i,\, \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \rangle \text{ for every } \boldsymbol{x} \in \mathbb{R}^{1+n} \ (i = 0, \ldots, m). \tag{22}$$

Then,

$$\langle \boldsymbol{Q}^i,\, \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \rangle = \bar{f}_i(\boldsymbol{x}) \text{ for every } \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in \mathbf{\Gamma}^{\mathcal{A}} \ (i = 0, \ldots, m). \tag{23}$$

Define $\mathbb{J} = \left\{ \boldsymbol{X} \in \mathbb{CPP}^{\mathcal{A}} : \langle \boldsymbol{Q}^i,\, \boldsymbol{X} \rangle = 0 \ (i = 1, \ldots, m) \right\}$. Then, we have that

$$\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J} \;=\; \left\{ \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T : \boldsymbol{x} \in \mathbb{R}^{1+n}_+,\ \bar{f}_i(\boldsymbol{x}) = 0 \ (i = 1, \ldots, m) \right\}.$$

Define $\boldsymbol{H}^0 \in \mathbb{S}^{\mathcal{A}}$ such that

$$\boldsymbol{H}^0 \;=\; \text{the symmetric matrix in } \mathbb{S}^{\mathcal{A}} \text{ whose elements are all 0 except the upper-leftmost element } H^0_{\boldsymbol{\alpha}^\omega \boldsymbol{\alpha}^\omega} \text{ that is set to 1.}$$

We then see that $\langle \boldsymbol{H}^0, \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \rangle = x_0^{2\omega}$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}$. It follows that $\boldsymbol{X} \in \boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}$ and $\langle \boldsymbol{H}^0, \boldsymbol{X} \rangle = 1$ (*i.e.*, $\boldsymbol{X} \in G(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, 1)$) iff there is an $\boldsymbol{x} \in \mathbb{R}^{1+n}_+$ such that

$$\boldsymbol{X} = \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T, \;\; \bar{f}_i(\boldsymbol{x}) = 0 \;(i = 1, \ldots, m) \text{ and } \;x_0^{2\omega} = 1.$$

Since $\boldsymbol{x} \in \mathbb{R}^{1+n}_+$ implies $x_0 \geq 0$, the last equality can be replaced by $x_0 = 1$. Therefore, a feasible solution $\boldsymbol{x} \in \mathbb{R}^{1+n}$ of POP (21) with the objective value $\bar{f}_0(\boldsymbol{x})$ corresponds to a feasible solution $\boldsymbol{X}$ of COP$(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0)$ with the objective value $\langle \boldsymbol{Q}^0, \boldsymbol{X} \rangle = \bar{f}_0(\boldsymbol{x})$ through the correspondence $\boldsymbol{x} \leftrightarrow \boldsymbol{X} = \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T$. Thus, POP (21) (hence POP (1)) is equivalent to COP$(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0)$ and $\zeta^* = \zeta(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0, 1)$.

5.4 Reformulation of COP$(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, \boldsymbol{Q}^0)$ into COP$(\mathbb{J}, \boldsymbol{Q}^0)$

We assume that POP (21) (hence (1)) is feasible, which implies $G(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}, 1) \neq \emptyset$. We also see that $\boldsymbol{0} \neq \boldsymbol{H}^0 \in (\boldsymbol{\Gamma}^{\mathcal{A}})^* \subset \mathbb{J}^*$. Hence Condition $\mathrm{I}_{\mathbb{J}}$ holds.

Now we focus on Conditions $0_{\mathbb{J}}$ and $\mathrm{II}_{\mathbb{J}}$. Define a sequence $\mathbb{J}_p \subset \mathbb{S}^{\mathcal{A}}$ $(p = 0, \ldots, m)$ by (16) with $\mathbb{K} = \boldsymbol{\Gamma}^{\mathcal{A}}$ and $\mathbb{J}_0 = \mathrm{co}\mathbb{K} = \mathbb{CPP}^{\mathcal{A}}$. Obviously, $\mathbb{J} = \mathbb{J}_m$. By Lemma 4.2, we know that $\mathbb{J}$ becomes a face of $\mathbb{CPP}^{\mathcal{A}}$ if $\boldsymbol{Q}^p$ is copositive on $\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}$ $(p = 1, \ldots, m)$. Thus, the copositivity of $\boldsymbol{Q}^p$ on $\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}$ $(p = 1, \ldots, m)$ can be characterized in terms of $f_i(\boldsymbol{w})$ and $\bar{f}_i(\boldsymbol{x})$ $(i = 0, \ldots, m)$.

Define

$$S_0 = \mathbb{R}^n_+, \quad S_p = \{\boldsymbol{w} \in S_{p-1} : f_p(\boldsymbol{w}) = 0\} \;\; (p = 1, \ldots, m), \tag{24}$$

$$\widetilde{S}_0 = \mathbb{R}^n_+, \quad \widetilde{S}_p = \left\{\boldsymbol{w} \in \widetilde{S}_{p-1} : \bar{f}_p(0, \boldsymbol{w}) = 0\right\} \;\; (p = 1, \ldots, m), \tag{25}$$

$$\overline{S}_0 = \mathbb{R}^{1+n}_+, \quad \overline{S}_p = \left\{\boldsymbol{x} \in \overline{S}_{p-1} : \bar{f}_p(\boldsymbol{x}) = 0\right\} \;\; (p = 1, \ldots, m). \tag{26}$$

By the definition of $\boldsymbol{\Gamma}^{\mathcal{A}}$ and (23), we observe that

$$\begin{aligned}
\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_0 &= \boldsymbol{\Gamma}^{\mathcal{A}} = \left\{\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T : \boldsymbol{x} \in \overline{S}_0\right\}, \\
\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_p &= \left\{\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in \mathbb{J}_{p-1} : \bar{f}_p(\boldsymbol{x}) = 0\right\} \\
&= \left\{\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T : \boldsymbol{x} \in \mathbb{R}^{1+n}_+, \; \bar{f}_i(\boldsymbol{x}) = 0 \; (i = 1, \ldots, p)\right\} \\
&= \left\{\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T : \boldsymbol{x} \in \overline{S}_p\right\} \;\; (p = 1, \ldots, m),
\end{aligned} \tag{27}$$

$$\begin{aligned}
G(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_p, \rho^{2\omega}) &= \left\{\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in \boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_p : \begin{array}{l} \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}_+, \\ x_0^{2\omega} = \rho^{2\omega} \end{array}\right\} \\
&= \left\{\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T : \boldsymbol{x} = (\rho, \boldsymbol{w}) \in \overline{S}_p\right\} \\
&\quad (\rho \geq 0, \; p = 0, \ldots, m).
\end{aligned} \tag{28}$$

Now, we are ready to prove the lemma which is used to establish the main theorems (Theorems 5.2 and 5.3) with Lemma 4.2 and 4.3.

Lemma 5.1. *Recall that* $\eta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) = \inf\{\langle \boldsymbol{Q}^p, \boldsymbol{X}\rangle : \boldsymbol{X} \in \mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}\}$. *We have that*

(i) $\eta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) = \inf\left\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x} \in \overline{S}_{p-1}\right\}$ *(*$p = 1, \ldots, m$*);*

(ii) $\zeta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) = \inf\left\{\bar{f}_p(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_{p-1}\right\}$ *(*$p = 1, \ldots, m$*);*

(iii) $\zeta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 1) = \inf\left\{f_p(\boldsymbol{w}) : \boldsymbol{w} \in S_{p-1}\right\}$ *(*$p = 1, \ldots, m$*);*

(iv) $\zeta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_m, \boldsymbol{Q}^0, 0) = \inf\left\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_m\right\}$.

Proof. The equality in (i) follows from (23) and (27). It follows from (23), (25) and (28) with $\rho = 0$ that

$$\begin{aligned}\zeta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) &= \inf\left\{\langle \boldsymbol{Q}^p,\ \boldsymbol{X}\rangle : \boldsymbol{X} = \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})^T \in G(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, 0)\right\} \\ &= \inf\left\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x} = (0, \boldsymbol{w}) \in \overline{S}_{p-1}\right\} \\ &= \inf\left\{\bar{f}_p(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_{p-1}\right\}.\end{aligned}$$

Thus we have shown (ii). For (iii), we see that

$$\begin{aligned}\zeta(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 1) &= \inf\left\{\langle \boldsymbol{Q}^p,\ \boldsymbol{X}\rangle : \boldsymbol{X} = \boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})(\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x}))^T \in G(\mathbf{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, 1)\right\} \\ &= \inf\left\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x} = (1, \boldsymbol{w}), \boldsymbol{w} \in \overline{S}_{p-1}\right\} \ \text{(by (23), (24) and (28))} \\ &= \inf\left\{f_p(\boldsymbol{w}) : \boldsymbol{w} \in S_{p-1}\right\}.\end{aligned}$$

(iv) follows from the same argument as the proof of (ii) with replacing $p-1$ by m and p by 0. ☐

We introduce the following conditions for the theorems below. Let $p \in \{1, \ldots, m\}$.

$$\inf\left\{\bar{f}_p(\boldsymbol{x}) : \boldsymbol{x} \in \bar{S}_{p-1}\right\} \geq 0, \tag{29}$$

$$\inf\left\{f_p(\boldsymbol{w}) : \boldsymbol{w} \in S_{p-1}\right\} \geq 0, \tag{30}$$

$$\inf\left\{\bar{f}_p(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_{p-1}\right\} \geq 0, \tag{31}$$

$$\inf\left\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_m\right\} \geq 0. \tag{32}$$

We note that (29), (31) and (32) depend on the choice of ω, while (30) is independent from the choice. But (30) depends on how an optimization problem is formulated by a POP of the form (1), as we shall see in Section 6.2.

Theorem 5.2. *Assume that* $\mathbb{J}_{p-1}$ *is a face of* $\mathbb{CPP}^{\mathcal{A}}$ *for some* $p \in \{1, \ldots, m\}$.

(i) If (29) holds, then $\mathbb{J}_p$ *is a face of* $\mathbb{J}_{p-1}$ *and a face of* $\mathbb{CPP}^{\mathcal{A}}$.

(ii) If (30) and (31) hold, then $\mathbb{J}_p$ *is a face of* $\mathbb{J}_{p-1}$ *and a face of* $\mathbb{CPP}^{\mathcal{A}}$.

Proof. (i) By (i) of Lemma 5.1, we know that $\eta(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p) \geq 0$. Hence, the assertion follows from Lemma 4.2.

(ii) Let $\mathbb{K} = \boldsymbol{\Gamma}^{\mathcal{A}}$. By (ii) and (iii) of Lemma 5.1, (30) and (31) are equivalent to $\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 1) \geq 0$ (*i.e.*, (19)) and $\zeta(\mathbb{K} \cap \mathbb{J}_{p-1}, \boldsymbol{Q}^p, 0) \geq 0$, respectively. Hence, the assertion follows from Lemmas 4.2 and 4.3. □

Theorem 5.3. *Assume that POP (1) is feasible and that $\mathbb{J} = \mathbb{J}_m$ is a face of $\mathbb{CPP}^{\mathcal{A}}$. Then $\zeta^* = \zeta(\mathbb{J}, \boldsymbol{Q}^0)$ iff*

$$\inf\left\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_m\right\} \geq 0 \ (\textit{i.e., } (32)) \ \textit{or} \ \zeta^* = -\infty \ \textit{holds.} \tag{33}$$

Proof. By (iv) of Lemma 5.1, (32) is equivalent to $\zeta(\boldsymbol{\Gamma}^{\mathcal{A}} \cap \mathbb{J}_m, \boldsymbol{Q}^0, 0) \geq 0$, *i.e.*, Condition $\mathrm{II}_{\mathbb{J}}$ with $\mathbb{K} = \boldsymbol{\Gamma}^{\mathcal{A}}$. We also know that $\zeta^* = \zeta(\boldsymbol{\Gamma}^{1+n} \cap \mathbb{J}_m, Q^0, 1)$. Hence (33) is equivalent to (10) in (iii) of Theorem 3.6. □

Next, we make some preparations to discuss sufficient conditions for (30), (31) and (32) to hold. We can represent each $f_i(\boldsymbol{w})$ as follows:

$$f_i(\boldsymbol{w}) \ = \ \hat{f}_i(\boldsymbol{w}) + \tilde{f}_i(\boldsymbol{w}), \ \deg \hat{f}_i(\boldsymbol{w}) < 2\omega \text{ and } \deg \tilde{f}_i(\boldsymbol{w}) = 2\omega$$

$(i = 0, 1, \ldots, m)$. If $\deg f_i(\boldsymbol{w}) < 2\omega$, we assume that $\tilde{f}_i(\boldsymbol{w}) \equiv 0$. Since $\bar{f}_i(0, \boldsymbol{w}) = \tilde{f}_i(\boldsymbol{w})$, we know that

$$\bar{f}_i(0, \boldsymbol{w}) \ = \ \begin{cases} \text{a degree } 2\omega \text{ homogeneous} & \\ \qquad\text{polynomial function} & \text{if } \deg f_i(\boldsymbol{w}) = 2\omega, \\ 0 & \text{otherwise, } \textit{i.e.}, \deg f_i(\boldsymbol{w}) < 2\omega, \end{cases}$$

$(i = 0, \ldots, m)$. This implies that $\widetilde{S}_p$ is a cone and that $\inf\left\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_m\right\}$ is either 0 or $-\infty$. Therefore, we can replace (31) and (32) by

$$\inf\left\{\bar{f}_p(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_{p-1}\right\} = 0 \text{ and } \inf\left\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_m\right\} = 0,$$

respectively.

We present some sufficient conditions for (30), (31) and (32) to hold.

Lemma 5.4. *Let $p \in \{1, \ldots, m\}$.*

(i) Assume that $f_p(\boldsymbol{w}) \geq 0$ for every $\boldsymbol{w} \in \mathbb{R}^n_+$. Then (30) and (31) hold.

(ii) If $\widetilde{S}_{p-1} = \{\boldsymbol{0}\}$ or $\deg f_p(\boldsymbol{w}) < 2\omega$, then (31) holds.

(iii) If $\widetilde{S}_m = \{\boldsymbol{0}\}$ or $\deg f_0(\boldsymbol{w}) < 2\omega$, then (32) holds.

Proof. The results in (ii) and (iii) are straightforward from the discussion above. So we only prove (i). Since $S_{p-1} \subset \mathbb{R}^n_+$, (30) follows. To show (31), assume on the contrary that there is a $\tilde{\boldsymbol{w}}$ such that $\bar{f}_{p-1}(0, \tilde{\boldsymbol{w}}) = \tilde{f}_{p-1}(\tilde{\boldsymbol{w}}) < 0$ for some $\tilde{\boldsymbol{w}} \in \widetilde{S}_{p-1} \subset \mathbb{R}^n_+$. Since $\deg \hat{f}_{p-1}(\boldsymbol{w}) < \deg \tilde{f}_{p-1}(\boldsymbol{w}) = 2\omega$, we have that

$$\lambda\tilde{\boldsymbol{w}} \in \mathbb{R}^n_+ \text{ and } f_{p-1}(\lambda\tilde{\boldsymbol{w}}) = \lambda^{2\omega}\left(\hat{f}_{p-1}(\lambda\tilde{\boldsymbol{w}})/\lambda^{2\omega} + \tilde{f}_{p-1}(\tilde{\boldsymbol{w}})\right) < 0$$

for a sufficiently large λ. This contradicts the assumption. $\square$

Obviously, if $f_p(\boldsymbol{w})$ is a sum of squares of polynomials or a polynomial with nonnegative coefficients, then $f_p(\boldsymbol{w}) \geq 0$ for every $\boldsymbol{w} \in \mathbb{R}^n_+$. Otherwise, the constraint $f_p(\boldsymbol{w}) = 0$ can be replaced by $f_p(\boldsymbol{w})^2 = 0$ (i.e., the polynomial $f_p(\boldsymbol{w})$ is replaced by $f_p(\boldsymbol{w})^2$), then (30) and (31) are attained. By (ii) of Lemma 5.4, we also know that Condition $\mathrm{II}_{\mathbb{J}}$ is satisfied if we take a positive integer ω such that $\deg f_0(\boldsymbol{w}) < 2\omega$. Thus, we can easily construct an equivalent convex COP reformation, $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$ of POP (1) in theory.

6 Applying Theorems 5.2 and 5.3 to two examples

We illustrate how the main theorems, Theorems 5.2 and 5.3 in Section 5, can be applied to QOPs and POPs with two examples. The first one is QOP (4) which has already been reduced to $\mathrm{COP}(\boldsymbol{\Gamma}^{1+n} \cap \mathbb{J}, \boldsymbol{Q}^0)$ for some cone $\mathbb{J} \subset \mathbb{CPP}^{1+n} = \mathrm{co}\boldsymbol{\Gamma}^{1+n}$ in Section 2.2. The second one is a POP with some complicated combinatorial constraints.

6.1 QOP (4) revisited

Since QOP (4) is a special case of POP (1), all discussions in Section 5 can be applied to QOP (4) if $\boldsymbol{u}^{\mathcal{A}}(\boldsymbol{x})$, $\boldsymbol{\Gamma}^{\mathcal{A}}$ and $\mathbb{CPP}^{\mathcal{A}}$ are replaced by $(1, x_1, \ldots, x_n)$, $\boldsymbol{\Gamma}^{1+n}$ and $\mathbb{CPP}^{1+n}$, respectively. In fact, we have already constructed a cone $\mathbb{J} \subset \mathbb{CPP}^{1+n} = \mathrm{co}\boldsymbol{\Gamma}^{1+n}$ and derived $\mathrm{COP}(\boldsymbol{\Gamma}^{1+n} \cap \mathbb{J}, \boldsymbol{Q}^0)$, which is equivalent to QOP (4), in the same way described in Section 5.3. We have mentioned there that $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$ provides a CPP reformulation of QOP (4) under conditions (7) and (8). In this section, we prove this fact by applying Theorems 5.2 and 5.3 .

Recall that $\boldsymbol{Q}^p$ has been chosen to satisfy $\bar{f}_p(\boldsymbol{x}) = \langle \boldsymbol{Q}^p, \boldsymbol{x}\boldsymbol{x}^T \rangle$ for every $\boldsymbol{x} \in \mathbb{R}^{1+n}$ with $\bar{f}_p(\boldsymbol{x})$ given in (5) $(p = 0, \ldots, m)$. With $\mathbb{K} = \boldsymbol{\Gamma}^{1+n}$, define $\mathbb{J}_p$, S_p and $\widetilde{S}_p$ by (16), (24) and (25) $(p = 0, \ldots, m)$, respectively. Obviously $\mathbb{J} = \mathbb{J}_m$. We then see that

$$\begin{aligned} \bar{f}_1(\boldsymbol{x}) &= (\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}x_0)^T(\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}x_0) \geq 0 \text{ for every } \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}_+, \\ \bar{f}_2(\boldsymbol{x}) &= \sum_{(j,k)\in I_{\mathrm{comp}}} w_j w_k \geq 0 \text{ for every } \boldsymbol{x} = (x_0, \boldsymbol{w}) \in \mathbb{R}^{1+n}_+. \end{aligned}$$

By (i) of Theorem 5.2, $\mathbb{J}_1$ and $\mathbb{J}_2$ are faces of $\mathbb{CPP}^{1+n}$. Now, we show that $\mathbb{J}_p$ is a face of $\mathbb{J}_{p-1}$ for $p \in \{3, \ldots, m\}$. Let $p \in \{3, \ldots, m\}$ be fixed. It follows from (7) that

$$\begin{aligned} &S_{p-1} \subset S_1 = \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{w} - \boldsymbol{b} = \boldsymbol{0}\right\} = L \subset \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : w_i \leq 1 \ (i = 1, \ldots, m-2)\right\}, \\ &\widetilde{S}_{p-1} \subset \widetilde{S}_1 = \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : \boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}\right\} = L_\infty \subset \left\{\boldsymbol{w} \in \mathbb{R}^n_+ : w_i = 0 \ \ (i = 1, \ldots, m-2)\right\}. \end{aligned}$$

We then see

$$\begin{aligned} f_p(\boldsymbol{w}) &= w_{p-2}(1 - w_{p-2}) \geq 0 \text{ for every } \boldsymbol{w} \in S_{p-1} \text{ (hence (30) holds)}, \\ \bar{f}_p(0, \boldsymbol{w}) &= w_{p-2}(0 - w_{p-2}) = 0 \text{ for every } \boldsymbol{w} \in \widetilde{S}_{p-1} \text{ (hence (31) holds)}. \end{aligned}$$

By (ii) of Theorem 5.2, $\mathbb{J}_p$ is a face of $\mathbb{CPP}^{\mathcal{A}}$. Thus we have shown that $\mathbb{J}_p$ is a face of $\mathbb{CPP}^{\mathcal{A}}$ for $p = 3, \ldots, m$. Therefore, we can conclude that $\mathbb{J} = \mathbb{J}_m$ is a face of $\mathbb{CPP}^{1+n}$.

By Theorem 5.3, (33) is a necessary and sufficient condition for $\zeta^* = \zeta(\mathbb{J}, \boldsymbol{Q}^0, 1)$. We show that the pair of (7) and (8) is a sufficient condition for (33) to hold. We see from conditions (7) and (8) that

$$\widetilde{S}_m \subset \widetilde{S}_2 \quad \subset \quad \widehat{S} \equiv \left\{ \boldsymbol{w} \in \mathbb{R}^n_+ : \begin{array}{l} \boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}, \ w_i = 0 \ (i = 1, \ldots, m-2) \\ w_j = 0 \text{ and } w_k = 0 \ ((j,k) \in I_{\text{comp}}) \end{array} \right\}.$$

Let $\bar{\boldsymbol{w}}$ be a fasible solution of QOP (4). Suppose that $\bar{f}_0(0, \widetilde{\boldsymbol{w}}) = \widetilde{\boldsymbol{w}}^T \boldsymbol{C} \widetilde{\boldsymbol{w}} < 0$ for some $\widetilde{\boldsymbol{w}} \in \widetilde{S}_m \subset \widehat{S}$. Then $\bar{\boldsymbol{w}} + \lambda \widetilde{\boldsymbol{w}}$ is a feasible solution of QOP (4) with the objective value $f_0(\bar{\boldsymbol{w}} + \lambda \widetilde{\boldsymbol{w}}) \to -\infty$ as $\lambda \to \infty$. Hence $\zeta^* = -\infty$. On the contrary, if there is no such a $\widetilde{\boldsymbol{w}} \in \widetilde{S}_m$, then $0 \leq \inf\{\bar{f}_0(0, \boldsymbol{w}) : \boldsymbol{w} \in \widetilde{S}_m\}$; hence (32) holds. Therefore, we have shown that the pair of (7) and (8) implies (33).

6.2 A set of complicated combinatorial conditions from [1]

We consider a problem of minimizing a polynomial function in $(w_1, \ldots, w_4)$ subjct to the following combinatorial conditions.

$$\left. \begin{array}{l} 0 \leq w_j \leq 1 \ (j = 1,2,3), \ w_4 \in \{0,1\}, \\ w_1 = 1 \text{ and/or } w_2 = 1, \ \textit{i.e.}, \ (1 - w_1)(1 - w_2) = 0, \\ w_3 = 0 \text{ and/or } w_1 + w_2 - w_3 = 0, \ \textit{i.e.}, \ w_3(w_1 + w_2 - w_3) = 0, \\ w_4 = 0 \text{ and/or } 2 - w_1 - w_2 - w_3 = 0, \ \textit{i.e.}, \ w_4(2 - w_1 - w_2 - w_3) = 0. \end{array} \right\} \quad (34)$$

To represent these conditions by polynomial equality constraints, we define 4 polynomial functions in $\boldsymbol{w} = (w_1, \ldots, w_8) \in \mathbb{R}^8$. Choose a positive integer ω not less than the half of the degree of the objective polynomial function. Define the 4 polynomial functions $f_i(\boldsymbol{w})$ in $\boldsymbol{w} = (w_1, \ldots, w_8) \in \mathbb{R}^8$ $(i = 1, \ldots, 4)$ by

$$\begin{array}{l} f_1(\boldsymbol{w}) = \displaystyle\sum_{k=1}^{4} (w_k + w_{k+4} - 1)^{2\omega}, \ f_2(\boldsymbol{w}) = w_4(1 - w_4) + (1 - w_1)(1 - w_2), \\ f_3(\boldsymbol{w}) = w_3(w_1 + w_2 - w_3) \text{ and } f_4(\boldsymbol{w}) = w_4(2 - w_1 - w_2 - w_3). \end{array}$$

Here $w_5, \ldots, w_8$ are slack variables for $w_1, \ldots, w_4$, respectively. Then the combinatorial constraints in w_1, w_2, w_3, w_4 above are satisfied iff $\boldsymbol{w} \in \mathbb{R}^8_+$ and $f_i(\boldsymbol{w}) = 0$ $(i = 1,2,3,4)$ for some w_5, w_6, w_7, w_8. We also set the objective polynomial function $f_0(\boldsymbol{w})$ in $\boldsymbol{w} \in \mathbb{R}^8$ by adding the dummy variables w_5, w_6, w_7, w_8 to the original one in $(w_1, \ldots, w_4)$. Thus the problem is formulated as POP (1) with $n = 8$ and $m = 4$. By taking $\omega \geq \lceil \deg f_0(\boldsymbol{w})/2 \rceil$, Theorems 5.2 and 5.3 can be applied to POP (1) as shown in the following.

Homogenizing the polynomial functions $f_i(\boldsymbol{w})$ $(i = 1, \ldots, 4)$ with degree 2ω, we obtain that

$$\begin{array}{rcl} \bar{f}_1(x_0, \boldsymbol{w}) & = & \sum_{k=1}^{4} (w_k + w_{k+4} - x_0)^{2\omega}, \\ \bar{f}_2(x_0, \boldsymbol{w}) & = & x_0^{2\omega-2} \left(w_4(x_0 - w_4) + (x_0 - w_1)(x_0 - w_2) \right), \\ \bar{f}_{3(}x_0, \boldsymbol{w}) & = & x_0^{2\omega-2} w_3(w_1 + w_2 - w_3), \\ \bar{f}_4(x_0, \boldsymbol{w}) & = & x_0^{2\omega-2} w_4(2x_0 - w_1 - w_2 - w_3). \end{array}$$

Then

$$
\begin{array}{rcl}
S_0 & = & \widetilde{S}_0 = \mathbb{R}^8_+, \\
f_1(\boldsymbol{w}) & = & \sum_{k=1}^4 (w_k + w_{k+4} - 1)^{2\omega} \geq 0 \text{ for every } \boldsymbol{w} \in S_0, \\
S_1 & = & \left\{ \boldsymbol{w} \in \mathbb{R}^8_+ : f_1(\boldsymbol{w}) = 0 \right\} \subset [0,1]^8, \\
f_2(\boldsymbol{w}) & = & w_4(1-w_4) + (1-w_1)(1-w_2) \geq 0 \text{ for every } \boldsymbol{w} \in [0,1]^8 \supset S_1, \\
S_2 & = & \{ \boldsymbol{w} \in S_1 : f_2(\boldsymbol{w}) = 0 \} = \{ \boldsymbol{w} \in S_1 : w_4 \in \{0,1\}, \ w_1 = 1 \text{ or } w_2 = 1 \}, \\
f_3(\boldsymbol{w}) & = & w_3(w_1 + w_2 - w_3) \geq 0 \text{ for every } \boldsymbol{w} \in S_2, \\
S_3 & = & \{ \boldsymbol{w} \in S_2 : f_3(\boldsymbol{w}) = 0 \} = \{ \boldsymbol{w} \in S_2 : \ w_3 = 0 \text{ or } w_1 + w_2 - w_3 = 0 \}, \\
f_4(\boldsymbol{w}) & = & w_4(2 - w_1 - w_2 - w_3) \geq 0 \text{ for every } \boldsymbol{w} \in S_3, \\
\bar{f}_1(0, \boldsymbol{w}) & = & \sum_{k=1}^4 (w_k + w_{k+4})^{2\omega} \geq 0 \text{ for every } \boldsymbol{w} \in \widetilde{S}_0, \\
\widetilde{S}_1 & = & \left\{ \boldsymbol{w} \in \mathbb{R}^8_+ : \bar{f}_1(0, \boldsymbol{w}) = 0 \right\} = \{\boldsymbol{0}\}, \ \widetilde{S}_2 = \widetilde{S}_3 = \widetilde{S}_4 = \{\boldsymbol{0}\}, \\
\bar{f}_p(0, \boldsymbol{w}) & \geq & 0 \text{ for every } \boldsymbol{w} \in \widetilde{S}_{p-1} = \{\boldsymbol{0}\} \ (p = 2,3,4), \\
\bar{f}_0(0, \boldsymbol{w}) & \geq & 0 \text{ for every } \boldsymbol{w} \in \widetilde{S}_4 = \{\boldsymbol{0}\}.
\end{array}
$$

Thus, we have confirmed that (30) and (31) hold for $p = 1, \ldots, 4$, and (32) holds with $m = 4$. By Theorems 5.2 and 5.3, $\mathrm{COP}(\mathbb{J}_4, \boldsymbol{Q}^0)$ provides a convex COP reformulation of POP (1) with $n = 8$ and $m = 4$.

If additional nonnegative variables w_9 and w_{10} are introduced and some complementarity conditions are used, then (34) can also be represented as a single equality constraint

$$
\begin{aligned}
&\sum_{k=1}^{5} (w_k + w_{k+4} - 1)^{2\omega} + (w_9 - w_1 - w_2 + w_3)^{2\omega} \\
&\qquad + (w_{10} + w_1 + w_2 + w_3 - 2)^{2\omega} + w_4 w_8 + w_5 w_6 + w_3 w_9 + w_4 w_{10} = 0.
\end{aligned}
$$

In this case, we can apply the discussion at the end of Section 5.4.

For given combinatorial conditions, there exist multiple ways of representing them with binary conditions and complementarity conditions. For example, binary conditions can be replaced by some complementarity conditions with slack variables. See [17] for more detailed discussions.

7 Concluding remarks

We have presented the theoretical aspects of the CPP reformulation of QOPs and its extension to POPs. To compute a lower bound for the optimal value of POP (1), numerically tractable relaxations of the problem are necessary. Suppose that QOP (4) is equivalently convexified to $\mathrm{COP}(\mathbb{J}, \boldsymbol{Q}^0)$ for some face $\mathbb{J}$ of $\mathbb{CPP}^{1+n}$ as presented in Section 6.1, where $\mathbb{J}$ is represented as in (3) with $\mathbb{K} = \boldsymbol{\Gamma}^{1+n}$ and $\mathrm{co}\mathbb{K} = \mathbb{CPP}^{1+n}$. Since $\mathbb{CPP}^{1+n}$ is contained in the DNN cone $\mathbb{S}^{1+n} \cap \mathbb{N}^{1+n}$, the CPP cone $\mathbb{CPP}^{1+n}$ can be relaxed to the DNN cone to obtain a numerically tractable DNN relaxation of QOP (4) so as to compute a lower bound of its optimal value ζ_{QOP}. The effectiveness of this approach combined with the Lagrangian-DNN relaxation technique and the bisection and projection (BP) algorithm was confirmed through numerical results in [5, 16, 19], where binary QOPs, max stable set problems, multi-knapsack QOPs, quadratic assignment problems were solved. The BP algorithm was originally designed to work effectively and efficiently for Lagrangian-DNN relaxation problems induced from CPP reformulations of a class of QOPs with linear equality, binary and complementarity constraints in [19]. In fact, it was shown in [5] that Lagrangian-DNN relaxation problems induced from the CPP reformulations of binary QOP instances from [27] clearly provided tighter lower bounds than DNN relaxation problems obtained from their standard SDP relaxations with replacing the SDP cone by the DNN cone.

The aforementioned method using the Lagrangian-DNN relaxation technique and the BP algorithm for QOPs was extended to a class of sparse POPs with binary, box and complementarity constraints in [17, 18]. Numerical results on instances from the class showed that accurate lower bounds of their optimal values were efficiently obtained by the method. Consequently, the theoretical study of the CPP reformulation of QOPs and its extensions to POPs are very important, not only for understanding of their theoretical features, but also for practical implementation. See [17, 18] for more details.

References

[1] N. Arima, S. Kim, and M. Kojima. A quadratically constrained quadratic optimization model for completely positive cone programming. *SIAM J. Optim.*, 23(4):2320–2340, 2013.

[2] N Arima, S. Kim, and M. Kojima. Extension of completely positive cone relaxation to polynomial optimization. *J. Optim. Theory Appl.*, 168:884–900, 2016.

[3] N. Arima, S. Kim, M. Kojima, and K. C. Toh. Lagrangian-conic relaxations, Part I: A unified framework and its applications to quadratic optimization problems. *Pacific J. of Optim.*, 14(1):161–192, 2018.

[4] N. Arima, S. Kim, M. Kojima, and K. C. Toh. Lagrangian-conic relaxations, Part II: Applications to polynomial optimization problems. *To appear in Pacific J. of Optim.*

[5] N. Arima, S. Kim, M. Kojima, and K.C. Toh. A robust Lagrangian-DNN method for a class of quadratic optimization problems. *Comput. Optim. Appl.*, 66(3):453–479, 2017.

[6] A. Berman, M. Dur, and N. Shaked-Monderer. Open problems in the theory of completely positive and copositive matrices. *The Electric Journal of Linear Algebra*, 29:46–58, 2015.

[7] A. Berman and N. Shaked-Monderer. *Compoletely Positive Matrices.* World Scientific, 2003.

[8] I. M. Bomze, J. Cheng, P. J. C. Dickinson, and A. Lisser. A fresh CP look at mixed-binary QPs: new formulations and relaxations. *Mathematical Programming*, 166:159–184, 2017.

[9] I. M. Bomze, M. Dür, E. de Klerk, C. Roos, A. Quist, and T. Terlaky. On copositive programming and standard quadratic optimization problems. *J. Global Optim.*, 18:301–320, 2000.

[10] I. M. Bomze and E. de Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming, j. global optim., 24 (2002), pp. 163–185. *J. Global Optim.*, 24:163–185, 2002.

[11] S. Burer. On the copositive representation of binary and continuous non-convex quadratic programs. *Math. Program.*, 120:479–495, 2009.

[12] E. de Klerk and D. V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM J. Optim.*, 12(875-892), 2002.

[13] P. J. C. Dickinson. Geometry of the copositive and completely positive cones. *J. Math. Anal. Appl.*, 380:377–395, 2011.

[14] P. J. C. Dickinson, G. Eichfelder, and J. Povh. Erratum to: "on the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets" [optim. letters, 2012]. *Optim. Lett.*, 7(1387-1397), 2013.

[15] D. Ge and Y. Ye. On doubly positive semidefinite programming relaxations. http://www.optimization-online.org/DB_HTML/2010/08/2709.html, August 2010.

[16] N. Ito, S. Kim, M. Kojima, A. Takeda, and K.C. Toh. BBCPOP: A sparse doubly nonnegative relaxation of polynomial optimization problems with binary, box and complementarity constraints. Technical Report arXiv:1804.00761, 2018.

[17] N. Ito, S. Kim, M. Kojima, A. Takeda, and K.C. Toh. Equivalences and differences in conic relaxations of combinatorial quadratic optimization problems. *Journal of Global Optimization*, 72(4):619–653, November 2018.

[18] S. Kim, M. Kojima, and K. C. Toh. Doubly nonnegative relaxations for quadratic and polynomial optimization problems with binary and box constraints. Research Rport B-483, Tokyo Institute of Technology, Department of Mathematical and Computing Sciences, Oh-Okayama, Meguro-ku, Tokyo 152-8552, July 2016.

[19] S. Kim, M. Kojima, and K. C. Toh. A Lagrangian-DNN relaxation: a fast method for computing tight lower bounds for a class of quadratic optimization problems. *Math. Program.*, 156:161–187, 2016.

[20] M. Kojima, S. Kim, and H. Waki. Sparsity in sums of squares of polynomials. *Mathematical Programming*, 103:45–62, 2005.

[21] J. B. Lasserre. Global optimization with polynomials and the problems of moments. *SIAM J. Optim.*, 11:796–817, 2001.

[22] J. Pena, J. C. Vera, and L. F. Zuluaga. Completely positive reformulations for polynomial optimization. *Math. Program.*, 151(2):405–431, July 2015.

[23] S. Poljak, F. Rendl, and H. Wolkowicz. A recipe for semidefinite relaxation for (0,1)-quadratic programming. *J. Global Optim.*, 7:51–73, 1995.

[24] J. Povh and F. Rendl. A copositive programming approach to graph partitioning. *SIAM J. Optim.*, 18:223–241, 2007.

[25] J. Povh and F. Rendl. Copositive and semidefinite relaxations of the quadratic assignment problem. *Discrete Optim.*, 6:231–241, 2009.

[26] M. Tanaka, K. Nakata, and H. Waki. Application of a facial reduction algorithm and an inexact primal-dual path-following mathod for doubly nonnegative relaxation for mixed binary nonconvex quadratic optimization problems. *Pacific J. of Optim.*, 8:699–724, 2012.

[27] A. Wiegele. Biq mac library. http://www.biqmac.uni-klu.ac.at/biqmaclib.html, 2007.

[28] A. Yoshise and Y. Matsukawa. On optimization over the doubly nonnegative cone. IEEE Multi-conference on Systems and Control, 2010.

[29] Q. Zhang. Completely positive cones: Are they facially exposed? *Linear Algebra and Its Applications*, 558:195–204, 2018.