A QUADRATICALLY CONSTRAINED QUADRATIC OPTIMIZATION MODEL FOR COMPLETELY POSITIVE CONE PROGRAMMING

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Abstract. We propose a class of quadratic optimization problems which can be reformulated by completely positive cone programming with the same optimal values. The objective function can be any quadratic form. The constraints of each problem are described in terms of quadratic forms with no linear terms, and all constraints are homogeneous equalities, except one inhomogeneous equality where a quadratic form is set to be a positive constant. For the equality constraints, "a hierarchy of copositvity" condition is assumed. This model is a generalization of the standard quadratic optimization problem of minimizing a quadratic form over the standard simplex, and covers many of the existing quadratic optimization problems studied for exact copositive cone and completely positive cone programming relaxations. In particular, it generalizes the recent results on quadratic optimization problems by Burer and the set-semidefinite representation by Eichfelder and Povh.

Key words. Copositve programming, quadratic optimization problem with quadratic constraints, a hierarchy of copositivity.

AMS subject classifications. 90C20, 90C25, 90C26

1. Introduction.

A nonconvex quadratic optimization problem (abbreviated as QOP) with quadratic equality and inequality constraints is known to be an NP hard problem. It may include binary variables, and covers many important combinatorial optimization problems such as max-cut problems, maximum stable set problems, and quadratic assignment problems. Semidefinite programming (abbreviated by SDP) relaxation techniques have been effectively used to compute bounds for its optimal value and approximate optimal solution [13, 14, 17, 18, 19, 23, 26, 27, 28].

Relaxations using copositive cones and completely positive cones have attracted a great deal of attention [3, 5, 6, 10, 11, 15, 21, 24, 25] in recent years. As a result, many theoretical properties of the copositive and completely positive cones are known [1, 4, 6, 8, 10], which can be used for developing effective relaxations. Despite numerical intractability of the copositive and completely positive cones used there, the study on the relaxations is popular since they provide much stronger relaxations than SDP relaxations. Relaxations using copositive cones and completely positive cones form linear optimization problems over closed convex cones, and they are indeed the same type of problems as linear programming problems, second order cone programming problems and SDP problems for which powerful primal-dual interior-point methods have been developed by [16, 20]. In addition, copositive and completely positive cones can be expressed in a very simple form, as in the case of the cone of positive semidefinite symmetric matrices. These facts have motivated us to study copositive and completely positive cone relaxations of QOPs as a solution method for QOPs. In particular, Burer [5] formulated the class of QOPs with linear constraints in both

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nonnegative continuous variables and binary variables as a linear optimization problem over a completely positive cone (abbreviated as CPP); more precisely, a QOP in the class has the same optimal objective value of its CPP relaxation. Eichfelder and Povh [11] extended Burer's results to a QOP with an additional constraint $\boldsymbol{u} \in D$ in its variable vector $\boldsymbol{u}$, where D is a closed (not necessarily convex) set. This paper presents a further extension of their results.

For related work, a QOP on the standard simplex was formulated as a CPP [2, 3]. The maximum stable set problem in [15], a graph tri-partitioning problem in [24], and the quadratic assignment problem [24], were considered and reduced to CPPs. More recently, general QOPs with quadratic constraints were represented as generalized CPPs in [7]. However, it is not well understood yet whether a general class of QOPs can be formulated as CPPs.

Our main purpose of this paper is to propose a new class of QOPs which have the same optimal objective values as their CPP relaxations. The proposed class not only covers the class of QOPs with linear constraints in both nonnegative continuous variables and binary variables, but also increases the prospects for formulating various QOPs in a more general form as CPPs. We will employ a QOP with a cone constraint $\boldsymbol{x} \in \mathbb{K}$ in its n-dimensional vector $\boldsymbol{x}$, where $\mathbb{K}$ is a closed (not necessarily convex) cone. Although working with $\boldsymbol{x}$ in the n-dimensional nonnegative orthant $\mathbb{R}^n_+$, the most important special case, was the starting point of this work, it immediately became clear that the generalization from the nonnegative orthant to a general closed cone is straightforward by just replacing $\mathbb{R}^n_+$ by $\mathbb{K}$ and modifying slightly. This generalization is described in the main results in Section 3 and their proofs in Section 4. In fact, $\mathbb{R}^n_+$ is a convex cone, but its convexity does not play any essential role.

This paper is organized as follows. In Section 2, we provide some notation and symbols for the subsequent sections, and introduce a standard form QOP whose exact CPP relaxation is a main subject of this paper. The QOP is described in terms of quadratic forms with no linear terms. The objective function is a quadratic form, and all constraints are homogeneous equalities in nonnegative variables, except one inhomogeneous equality where a quadratic form is set to be a positive constant. We show how a general QOP with a constraint $\boldsymbol{u} \in D$ in its variable vector $\boldsymbol{u}$ can be described in the standard form QOP. We also introduce a CPP relaxation of the standard form QOP.

In Section 3, we build a hierarchical structure into the constraint of the QOP. Two sets of conditions, simple ones in Section 3.1 and general ones in Section 3.2, are imposed on the hierarchically structured constraint to ensure that the QOP and its CPP relaxation have an equivalent optimal value. Among those conditions, "a hierarchy of copositivity" plays an essential role, which may be regarded as an extension of a one step copositivity condition in [5]. Section 3.1 deals with a simple case where a stronger and simple set of conditions on the compactness of the feasible region of the QOP is assumed. The main result (Theorem 3.2) is a special case of Theorem 3.5 in Section 3.2, where a similar result is established under a general and weaker set of conditions. The simple arguments in Section 3.1 may facilitate understanding of the arguments in Section 3.2.

Section 4 is devoted to proofs of the lemmas in Section 3.2. Four examples are presented in Section 5 to show that they can be reduced to the standard form QOP that satisfies either the simple set of conditions or the general set of conditions. Section 5.1 includes a QOP with linear equality constraints in nonnegative continuous variables and binary variables, and an additional constraint $\boldsymbol{u} \in D$ in its variable vector $\boldsymbol{u}$. This type of QOPs was studied in [11]. The last two examples demonstrate that the standard form QOP satisfying the general set of conditions can cover new types of QOPs. In Section 6, the concluding remarks are included.

2. Preliminaries.

2.1. Notation and symbols. We use the following notation and symbols throughout the paper.

$$
\begin{aligned}
\mathbb{R}^n &= \text{the space of } n\text{-dimensional column vectors},\\
\mathbb{R}^n_+ &= \text{the nonnegative orthant of } \mathbb{R}^n,\\
\mathbb{K} &= \text{a closed (not necessarily convex) cone in } \mathbb{R}^n,\\
\mathbb{S}^n &= \text{the space of } n \times n \text{ symmetric matrices},\\
\mathbb{S}^n_+ &= \text{the cone of } n \times n \text{ symmetric positive semidefinite matrices},\\
\mathbb{N} &= \text{the cone of } n \times n \text{ symmetric nonnegative matrices},\\
\mathbb{C}_{\mathbb{K}} &= \left\{ \boldsymbol{A} \in \mathbb{S}^n : \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in \mathbb{K} \right\} \text{ (a generalized copositive cone)},\\
\mathbb{C}^*_{\mathbb{K}} &= \left\{ \sum_{i=1}^{r} \boldsymbol{x}_j\boldsymbol{x}_j^T : \boldsymbol{x}_j \in \mathbb{K} \ (j = 1, 2, \ldots, r) \text{ for some } r \geq 1 \right\}\\
&\quad \text{(a generalized completely positive cone).}
\end{aligned}
$$

We know by Corollary 1.5 of [11] that $\mathbb{C}^*_{\mathbb{K}}$ is a closed convex cone and that $\mathbb{C}_{\mathbb{K}}$ and $\mathbb{C}^*_{\mathbb{K}}$ are dual of each other in $\mathbb{S}^n$:

$$
\begin{aligned}
\mathbb{C}^*_{\mathbb{K}} &= \{ \boldsymbol{X} \in \mathbb{S}^n : \boldsymbol{A} \bullet \boldsymbol{X} \geq 0 \text{ for every } \boldsymbol{A} \in \mathbb{C}_{\mathbb{K}} \},\\
\mathbb{C}_{\mathbb{K}} &= \{ \boldsymbol{A} \in \mathbb{S}^n : \boldsymbol{A} \bullet \boldsymbol{X} \geq 0 \text{ for every } \boldsymbol{X} \in \mathbb{C}^*_{\mathbb{K}} \}.
\end{aligned}
$$

Here $\boldsymbol{A} \bullet \boldsymbol{X}$ denotes the inner product $\sum_{i=1}^n \sum_{j=1}^n A_{ij}X_{ij}$ of $\boldsymbol{A} \in \mathbb{S}^n$ and $\boldsymbol{X} \in \mathbb{S}^n$. If we take $\mathbb{K} = \mathbb{R}^n_+$, $\mathbb{C}_{\mathbb{K}}$ and $\mathbb{C}^*_{\mathbb{K}}$ are known as the copositive cone and the completely positive cone, which will be simply denoted by $\mathbb{C}$ and $\mathbb{C}^*$, respectively. If $\mathbb{K} = \mathbb{R}^n$, both $\mathbb{C}_{\mathbb{K}}$ and $\mathbb{C}^*_{\mathbb{K}}$ coincide with $\mathbb{S}^n_+$. We have the relation $\mathbb{C}^* \subset \mathbb{S}^n_+ \bigcap \mathbb{N} \subset \mathbb{S}^n_+ \subset \mathbb{S}^n_+ + \mathbb{N} \subset \mathbb{C}$.

For $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T$ denotes the transposition of $\boldsymbol{x}$, and $\boldsymbol{x}^T$ is an n-dimensional row vector. We use notation $(\boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}^{m+n}$ for the $(m + n)$-dimensional column vector consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{s} \in \mathbb{R}^n$. The quadratic form $\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ associated with a matrix $\boldsymbol{Q} \in \mathbb{S}^n$ is represented as $\boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T$ for every $\boldsymbol{x} \in \mathbb{R}^n$. In the subsequent discussions, $\boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T$ is used to suggest that $\boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T$ with $\boldsymbol{x} \in \mathbb{K}$ is relaxed to $\boldsymbol{Q} \bullet \boldsymbol{X}$ with $\boldsymbol{X} \in \mathbb{C}^*_{\mathbb{K}}$.

For each subset U of $\mathbb{S}^n$, conv U denotes the convex hull of U, cl U the closure of U, and cone U the cone generated U; cone $U = \{\mu \boldsymbol{X} : \boldsymbol{X} \in U,\ \mu \geq 0\}$.

2.2. A standard form QOP and its CPP relaxation. Let $\rho > 0$, $\boldsymbol{Q} \in \mathbb{S}^n$ and $\boldsymbol{H}_k \in \mathbb{S}^n$ $(k = 0, 1, 2, \ldots, p)$. For the discussion on QOP, we consider a QOP of the form

$$
\begin{array}{ll}
\text{minimize} & \boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T\\
\text{subject to} & \boldsymbol{x} \in \mathbb{K},\ \boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T = \rho,\ \boldsymbol{H}_k \bullet \boldsymbol{x}\boldsymbol{x}^T = 0 \ (k = 1, 2, \ldots, p).
\end{array}
$$

Let

$$
\widetilde{G}_p = \left\{ \boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{K},\ \boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T = \rho,\ \boldsymbol{H}_k \bullet \boldsymbol{x}\boldsymbol{x}^T = 0 \ (k = 1, 2, \ldots, p) \right\}.
$$

Then, the QOP is rewritten as

$$
\text{minimize} \quad \boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T \quad \text{subject to} \quad \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_p. \tag{2.1}
$$

The important features of QOP (2.1) are: (i) the objective and constraint functions are all represented in terms of quadratic forms and (ii) the constraints are homogeneous equalities, except one nonhomogeneous equality where a quadratic form is set to a positive number. These two features play an essential role for the discussions in the next section.

We show that QOP (2.1) represents fairly general quadratic optimization problems. Let D be an arbitrary closed subset of $\mathbb{R}^n$. Consider a general QOP of the form

$$\begin{array}{lll} \text{(2.2)} & \text{minimize} & \boldsymbol{u}^T\boldsymbol{Q}_0\boldsymbol{u} + 2\boldsymbol{c}_0^T\boldsymbol{u} + \gamma_0 \\ & \text{subject to} & \boldsymbol{u}^T\boldsymbol{Q}_k\boldsymbol{u} + 2\boldsymbol{c}_k^T\boldsymbol{u} + \gamma_k = 0 \ (k = 1, 2, \ldots, p), \ \boldsymbol{u} \in D, \end{array}$$

where $\boldsymbol{Q}_k \in \mathbb{S}^m$, $\boldsymbol{c}_k \in \mathbb{R}^m$ and $\gamma_k \in \mathbb{R}$ $(k = 0, 1, \ldots, p)$. We embed the closed set $D \subset \mathbb{R}^m$ into the higher dimensional space $\mathbb{R}^{1+m}$ by letting $\mathbb{K}$ be the closure of $\left\{(u_0, u_0\boldsymbol{u}) \in \mathbb{R}^{1+m} : u_0 \geq 0, \ \boldsymbol{u} \in D\right\}$. Then, we can rewrite QOP (2.2) as

$$\begin{array}{ll} \text{minimize} & \boldsymbol{u}^T\boldsymbol{Q}_0\boldsymbol{u} + 2u_0\boldsymbol{c}_0^T\boldsymbol{u} + \gamma_0 u_0^2 \\ \text{subject to} & (u_0, \boldsymbol{u}) \in \mathbb{K}, \ u_0^2 = 1, \ \boldsymbol{u}^T\boldsymbol{Q}_k\boldsymbol{u} + 2u_0\boldsymbol{c}_k^T\boldsymbol{u} + \gamma_k u_0^2 = 0 \ (k = 1, 2, \ldots, p). \end{array}$$

By definition, $\mathbb{K}$ is a closed cone in $\mathbb{R}^{1+m}$, and not necessarily convex. Now, the objective and constraint functions are represented in quadratic forms of $(u_0, \boldsymbol{u}) \in \mathbb{R}^{1+m}$ with no linear nor constant terms. If we let

$$n = 1 + m, \ \rho = 1, \ \boldsymbol{x} = \begin{pmatrix} u_0 \\ \boldsymbol{u} \end{pmatrix} \in \mathbb{R}^n, \ \boldsymbol{Q} = \begin{pmatrix} \gamma_0 & \boldsymbol{c}_0^T \\ \boldsymbol{c}_0 & \boldsymbol{Q}_0 \end{pmatrix} \in \mathbb{S}^n,$$
$$\boldsymbol{H}_0 = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}^n, \ \boldsymbol{H}_k = \begin{pmatrix} \gamma_k & \boldsymbol{c}_k^T \\ \boldsymbol{c}_k & \boldsymbol{Q}_k \end{pmatrix} \in \mathbb{S}^n \ (k = 1, 2, \ldots, p),$$

we can reduce a general QOP in the form of (2.2) to QOP (2.1).

Our CPP relaxation of QOP (2.1) is obtained by replacing $\boldsymbol{x}\boldsymbol{x}^T$ with $\boldsymbol{X} \in \mathbb{C}^*_{\mathbb{K}}$.

$$\text{(2.3)} \qquad \text{minimize} \quad \boldsymbol{Q} \bullet \boldsymbol{X} \quad \text{subject to} \quad \boldsymbol{X} \in \widehat{G}_p,$$

where

$$\widehat{G}_p = \left\{\boldsymbol{X} \in \mathbb{C}^*_{\mathbb{K}} : \boldsymbol{H}_0 \bullet \boldsymbol{X} = \rho, \ \boldsymbol{H}_k \bullet \boldsymbol{X} = 0 \ (k = 1, 2, \ldots, p)\right\}.$$

Our main results presented in Section 3 assert that CPP (2.3) has the same optimal objective value as QOP (2.1) under certain assumptions.

Remark 2.1. A special case of QOP (2.2) was studied in [11] where Eichfelder and Povh extended Burer's result [5] on QOPs with linear constraints in both continuous nonnegative variables and binary variables to QOPs with an additional nonconvex constraint $\boldsymbol{x} \in D$. In particular, more general cones of symmetric matrices than $\mathbb{C}_{\mathbb{K}}$ and $\mathbb{C}^*_{\mathbb{K}}$ were introduced in [11]. Let K be an arbitrary nonempty subset of $\mathbb{R}^n$. Then, $C_K = \left\{\boldsymbol{A} \in \mathbb{S}^n : \boldsymbol{A} \bullet \boldsymbol{x}\boldsymbol{x}^T \geq 0 \text{ for all } \boldsymbol{x} \in K\right\}$ was called K-semidefinite (or set-semidefinite) cone. They reduced this particular special case of QOP (2.2) to a linear optimization problem over the dual of $C_{1\times D}$, where $1 \times D = \{(1, \boldsymbol{d}) : \boldsymbol{d} \in D\}$. See also Remark 5.1. One contribution of this paper is that the special case is extended to a QOP (2.1), which is more general than QOP (2.2).

3. Main results. In order to describe the assumptions on QOP (2.1), we construct a hierarchy into its constraint set $\widetilde{G}_p$. Define subsets $\widetilde{G}_\ell$ of $\mathbb{C}^*_{\mathbb{K}}$ ($\ell = 0, 1, 2, \ldots, p$) recursively by

$$
\begin{array}{rcl}
\widetilde{G}_0 & = & \left\{ \boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{K},\ \boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T = \rho \right\}, \\
\widetilde{G}_\ell & = & \left\{ \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1} : \boldsymbol{H}_\ell \bullet \boldsymbol{x}\boldsymbol{x}^T = 0 \right\} \\
 & = & \left\{ \boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{K},\ \boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T = \rho \text{ and } \boldsymbol{H}_k \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\ (k = 1, 2, \ldots, \ell) \right\} \\
 & & (\ell = 1, \ldots, p).
\end{array} \tag{3.1}
$$

Since the objective function of QOP (2.1) is linear in $\boldsymbol{x}\boldsymbol{x}^T$, QOP (2.1) is equivalent to

$$
\text{minimize} \quad \boldsymbol{Q} \bullet \boldsymbol{X} \quad \text{subject to} \quad \boldsymbol{X} \in \text{cl conv } \widetilde{G}_p. \tag{3.2}
$$

More precisely, $\inf \left\{ \boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_p \right\} = \inf \left\{ \boldsymbol{Q} \bullet \boldsymbol{X} : \boldsymbol{X} \in \text{cl conv } \widetilde{G}_p \right\}$. We impose some conditions on $\boldsymbol{H}_0$, $\boldsymbol{H}_\ell$ and $\widetilde{G}_{\ell-1}$ ($\ell = 1, 2, \ldots, p$) that ensure the identity $\widehat{G}_p = \text{cl conv } \widetilde{G}_p$. Then, QOP (3.2) and CPP (2.3) have an equivalent optimal value.

We describe a simple case in Section 3.1 and a general case under weaker assumptions in Section 3.2. The simple case may be regarded as a special case of the general case. The discussion in Section 3.1 is intended to help the readers understand slightly elaborate arguments in Section 3.2.

3.1. A simple case. We impose the following conditions on $\boldsymbol{H}_0$, $\boldsymbol{H}_\ell$ and $\widetilde{G}_{\ell-1}$ ($\ell = 1, 2, \ldots, p$) throughout Section 3.1.

(A) $\mathbb{R} \ni \rho > 0$ and $\boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T > 0$ for every nonzero $\boldsymbol{x} \in \mathbb{K}$.

($\widetilde{\text{B}}$) For every $\ell = 1, 2, \ldots, p$,

$$
\boldsymbol{H}_\ell \bullet \boldsymbol{x}\boldsymbol{x}^T \geq 0 \text{ if } \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1}. \tag{3.3}
$$

Remark 3.1. We discuss the relationship between the conditions above and the two conditions in the middle of page 488 of [5], which were said to be essential to prove an equivalence of a QOP with linear equality constraints in nonnegative continuous variables and binary variables to its CPP relaxation. Condition ($\widetilde{\text{B}}$) may be regarded as an extension of the first condition to our hierarchical QOP model, and Condition (D) in Section 3.2 is a generalization of the second condition. It was mentioned in Section 3.2 of [5] that the complementarity constraint $x_i \geq 0,\ x_j \geq 0$ and $x_i x_j = 0$ satisfies the first condition, thus, complementarity constraints could be added to their QOP if the constraints satisfies the second condition. This also applies to our hierarchical QOP model.

Clearly, $\widetilde{G}_0$ is bounded by (A), and so are $\widetilde{G}_\ell$ ($\ell = 1, 2, \ldots, p$) since $\widetilde{G}_\ell \subset \widetilde{G}_0$ ($\ell = 1, 2, \ldots, p$). If $\boldsymbol{H}_\ell$ is positive definite, then ($\widetilde{\text{B}}$) is trivially satisfied. In this case, however, $\widetilde{G}_\ell = \emptyset$ ($\ell = 1, \ldots, p$) since $\boldsymbol{0} \notin \widetilde{G}_0$. In general, low rank matrices are used for $\boldsymbol{H}_\ell$ ($\ell = 1, \ldots, p$). Condition (A) requires $\boldsymbol{H}_0$ to be chosen from the interior of $\mathbb{C}_{\mathbb{K}}$. Let $\ell \in \{1, 2, \ldots, p\}$. If $\widetilde{G}_\ell$ is nonempty, then (3.3) means that $\{\boldsymbol{X} \in \mathbb{S}^n : \boldsymbol{H}_\ell \bullet \boldsymbol{X} = 0\}$ forms a supporting hyperplane of $\widetilde{G}_{\ell-1}$ at every $\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_\ell$.

In the hierarchical construction of $\widetilde{G}_\ell$ ($\ell = 0, 1, 2, \ldots, p$) in (3.1), a single homogeneous equality $\boldsymbol{H}_\ell \bullet \boldsymbol{x}\boldsymbol{x}^T = 0$ is added at each level $\ell \geq 1$. We can extend this construction so that multiple homogeneous equalities are added at each level $\ell \geq 1$. Suppose that

$$
\widetilde{G}_\ell = \left\{ \boldsymbol{x} \in G_{\ell-1} : \boldsymbol{H}_{\ell i} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\ (i = 1, 2, \ldots, i_\ell) \right\}\ (\ell = 1, \ldots, p),
$$

where $\boldsymbol{H}_{\ell i} \in \mathbb{S}^n$ $(i = 1, 2, \ldots, i_\ell, \ell = 1, 2, \ldots, p)$. In this case, $(\widetilde{\mathrm{B}})$ is replaced by

$(\widetilde{\mathrm{B}})$' For every $\ell = 1, 2, \ldots, p$, $\boldsymbol{H}_{\ell i} \bullet \boldsymbol{x}\boldsymbol{x}^T \geq 0$ $(i = 1, 2, \ldots, i_\ell)$ if $\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1}$.

But, under Condition $(\widetilde{\mathrm{B}})$', we see that $\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1}$ and $\boldsymbol{H}_{\ell i} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0$ $(i = 1, 2, \ldots, i_\ell)$ if and only if $\boldsymbol{x} \in \widetilde{G}_{\ell-1}$ and $\left(\sum_{i=1}^{i_\ell} \boldsymbol{H}_{\ell i}\right) \bullet \boldsymbol{x}\boldsymbol{x}^T = 0$. As a result, if we let $\boldsymbol{H}_\ell = \sum_{i=1}^{i_\ell} \boldsymbol{H}_{\ell i}$, then $\widetilde{G}_\ell$ can be rewritten as in (3.1) with a single homogeneous equality added at each level ℓ. We emphasize that this technique is effective in reducing the number of equality constraints in QOP (2.1) and its CPP relaxation (2.3). In particular, it is shown in Section 5.1 that a QOP with linear constraints in both continuous nonnegative variables and binary variables is formulated as a QOP with three equality constraints.

Now, we introduce the completely positive cone relaxations $\widehat{G}_\ell$ of $\widetilde{G}_\ell$ $(\ell = 0, 1, 2, \ldots, p)$ by

$$\begin{array}{rcl} \widehat{G}_0 & = & \left\{\boldsymbol{X} \in \mathbb{C}^*_{\mathbb{K}} : \boldsymbol{H}_0 \bullet \boldsymbol{X} = \rho\right\}, \\ \widehat{G}_\ell & = & \left\{\boldsymbol{X} \in \widehat{G}_{\ell-1} : \boldsymbol{H}_\ell \bullet \boldsymbol{X} = 0\right\} \\ & = & \left\{\boldsymbol{X} \in \mathbb{C}^*_{\mathbb{K}} : \boldsymbol{H}_0 \bullet \boldsymbol{X} = \rho \text{ and } \boldsymbol{H}_k \bullet \boldsymbol{X} = 0 \ (k = 1, 2, \ldots, \ell)\right\} \\ & & (\ell = 1, 2, \ldots, p). \end{array} \tag{3.4}$$

Theorem 3.2. *Assume Conditions (A) and* $(\widetilde{\mathrm{B}})$*. Then,* conv $\widetilde{G}_\ell = \widehat{G}_\ell$ $(\ell = 0, 1, \ldots, p)$.

Proof. Let $\ell \in \{0, 1, \ldots, p\}$. Since $\widetilde{G}_\ell \subset \widehat{G}_\ell$ and $\widehat{G}_\ell$ is convex, conv $\widetilde{G}_\ell \subset \widehat{G}_\ell$ follows. We apply the induction on $\ell = 0, 1, \ldots, p$ to prove $\widehat{G}_\ell \subset$ conv $\widetilde{G}_\ell$.

Let $\ell = 0$. Suppose that $\boldsymbol{X} \in \widehat{G}_0$. Then $\boldsymbol{X} \neq \boldsymbol{O}$ and there exist nonzero $\boldsymbol{x}_i\boldsymbol{x}_i^T \in \widetilde{G}_0$ $(i = 1, 2, \ldots, r)$ such that

$$\boldsymbol{X} = \sum_{i=1}^r \boldsymbol{x}_i\boldsymbol{x}_i^T, \quad \boldsymbol{H}_0 \bullet \left(\sum_{i=1}^r \boldsymbol{x}_i\boldsymbol{x}_i^T\right) = \rho.$$

Let $\lambda_i = \boldsymbol{H}_0 \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T/\rho$ $(i = 1, 2, \ldots, r)$, which are positive by (A), and let

$$\boldsymbol{y}_i = \boldsymbol{x}_i/\sqrt{\lambda_i} \in \mathbb{K} \ (i = 1, 2, \ldots, r), \ \boldsymbol{Y}_i = \boldsymbol{y}_i\boldsymbol{y}_i^T \ (i = 1, 2, \ldots, r),$$

Then,

$$\boldsymbol{H}_0 \bullet \boldsymbol{Y}_i = \boldsymbol{H}_0 \bullet (\boldsymbol{x}_i/\sqrt{\lambda_i})(\boldsymbol{x}_i/\sqrt{\lambda_i})^T = \frac{\boldsymbol{H}_0 \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T}{\lambda_i} = \rho \ (i = 1, 2, \ldots, r).$$

As a result, $\boldsymbol{Y}_i \in \widetilde{G}_0$ $(i = 1, 2, \ldots, r)$. Furthermore, we see that

$$\begin{array}{c} \boldsymbol{X} = \displaystyle\sum_{i=1}^r \lambda_i(\boldsymbol{x}_i/\sqrt{\lambda_i})(\boldsymbol{x}_i/\sqrt{\lambda_i})^T = \sum_{i=1}^r \lambda_i \boldsymbol{y}_i\boldsymbol{y}_i^T = \sum_{i=1}^r \lambda_i \boldsymbol{Y}_i, \\ \displaystyle\sum_{i=1}^r \lambda_i = \sum_{i=1}^r \boldsymbol{H}_0 \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T/\rho = 1, \ \lambda_i > 0 \ (i = 1, 2, \ldots, r). \end{array}$$

Therefore, we have shown that $\boldsymbol{X}$ is a convex combination of $\boldsymbol{Y}_i \in \widetilde{G}_0$ $(i = 1, 2, \ldots, r)$.

Now, let $\ell \geq 1$ and assume the inclusion relations $\widehat{G}_k \subset \text{conv}\ \widetilde{G}_k$ $(k = 1, 2, \ldots, \ell - 1)$ to prove the relation $\widehat{G}_\ell \subset \text{conv}\ \widetilde{G}_\ell$. Suppose that $\boldsymbol{X} \in \widehat{G}_\ell$. It follows from $\widehat{G}_\ell \subset \widehat{G}_{\ell-1} \subset \text{conv}\ \widetilde{G}_{\ell-1}$ that $\boldsymbol{X} \in \text{conv}\ \widetilde{G}_{\ell-1}$. Hence, there exist $\boldsymbol{x}_i\boldsymbol{x}_i^T \in \widetilde{G}_{\ell-1}$ and $\lambda_i > 0$ $(i = 1, 2, \ldots, r)$ such that

$$\boldsymbol{X} = \sum_{i=1}^{r} \lambda_i \boldsymbol{x}_i\boldsymbol{x}_i^T,\ \sum_{i=1}^{r} \lambda_i = 1.$$

By Condition $(\widetilde{\text{B}})$, $\boldsymbol{H}_\ell \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T \geq 0$ $(i = 1, 2, \ldots, r)$. On the other hand, we know $\boldsymbol{X} = \sum_{i=1}^{r} \lambda_i \boldsymbol{x}_i\boldsymbol{x}_i^T \in \widehat{G}_\ell$. Thus, $0 = \boldsymbol{H}_\ell \bullet \boldsymbol{X} = \sum_{i=1}^{r} \lambda_i \left(\boldsymbol{H}_\ell \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T\right)$. By the inequalities $\boldsymbol{H}_\ell \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T \geq 0$ and $\lambda_i > 0$, we obtain that $\boldsymbol{H}_\ell \bullet \boldsymbol{x}_i\boldsymbol{x}_i^T = 0$, which, with $\boldsymbol{x}_i\boldsymbol{x}_i^T \in \widetilde{G}_{\ell-1}$, implies that $\boldsymbol{x}_i\boldsymbol{x}_i^T \in \widetilde{G}_\ell$ $(i = 1, 2, \ldots, r)$. Since $\boldsymbol{X}$ is a convex combination of $\boldsymbol{x}_i\boldsymbol{x}_i^T \in \widetilde{G}_\ell$ $(i = 1, 2, \ldots, r)$, we have shown that $\boldsymbol{X} \in \text{conv}\ \widetilde{G}_\ell$.

Theorem 3.2 ensures that conv $\widetilde{G}_\ell$ $(\ell = 0, 1, \ldots, p)$ are closed and that QOP (3.2) and CPP (2.3) are equivalent under Conditions (A) and $(\widetilde{\text{B}})$. In Theorem 3.2, we can replace Condition $(\widetilde{\text{B}})$ with

$(\widehat{\text{B}})$ For every $\ell = 1, 2, \ldots, p$, $\boldsymbol{H}_\ell \bullet \boldsymbol{X} \geq 0$ if $\boldsymbol{X} \in \widehat{G}_{\ell-1}$.

In fact, we see that $(\widehat{\text{B}})$ implies $(\widetilde{\text{B}})$ since $\widetilde{G}_{\ell-1} \subset \widehat{G}_{\ell-1}$, and that if $(\widetilde{\text{B}})$ holds then conv $\widetilde{G}_{\ell-1} = \widehat{G}_{\ell-1}$ by the theorem.

3.2. A general case under weaker conditions. When a given QOP is formulated in the form of (2.1) by constructing the hierarchy of its feasible region with $\widetilde{G}_\ell$ $(\ell = 0, 1, \ldots, p)$, Condition (A) on the boundedness of $\widetilde{G}_0$ may prevent a straightforward reformulation, even when the resulting feasible region $\widetilde{G}_p$ is bounded. See Section 5.1 for such an example. Motivated by this observation, we deal with the problems where $\widetilde{G}_p$ can be unbounded in this subsection.

Let

$$\begin{aligned} \widetilde{L}_0 &= \left\{\boldsymbol{d}\boldsymbol{d}^T : \boldsymbol{d} \in \mathbb{K},\ \boldsymbol{H}_0 \bullet \boldsymbol{d}\boldsymbol{d}^T = 0\right\}, \\ \widetilde{L}_\ell &= \left\{\boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_{\ell-1} : \boldsymbol{H}_\ell \bullet \boldsymbol{d}\boldsymbol{d}^T = 0\right\}\ (\ell = 1, 2, \ldots, p). \end{aligned} \tag{3.5}$$

Let $\ell \in \{0, 1, \ldots, p\}$. We call $\boldsymbol{d}\boldsymbol{d}^T \in \mathbb{C}^*_{\mathbb{K}}$ an *asymptotic unbounded direction* of $\widetilde{G}_\ell$ if there is a sequence

$$\left\{(\mu^s, \boldsymbol{u}^s(\boldsymbol{u}^s)^T) \in \mathbb{R}_+ \times \widetilde{G}_\ell : s = 1, 2, \ldots\right\}$$

such that $\|\boldsymbol{u}^s\| \to \infty$ and $(\sqrt{\mu^s}, \sqrt{\mu^s}\boldsymbol{u}^s) \to (0, \boldsymbol{d})$ (or equivalently $(\mu^s, \mu^s\boldsymbol{u}^s(\boldsymbol{u}^s)^T) \to (0, \boldsymbol{d}\boldsymbol{d}^T)$) as $s \to \infty$. We can prove that if $\boldsymbol{d}\boldsymbol{d}^T \in \mathbb{C}^*_{\mathbb{K}}$ is an asymptotic unbounded direction of $\widetilde{G}_\ell$, then $\boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_\ell$. But, the converse is not necessarily true in general, even when $\mathbb{K} = \mathbb{R}^n_+$. We show such examples below and in Section 5.4. It will be required that $\widetilde{L}_p \backslash \{\boldsymbol{O}\}$ coincide with the set of asymptotic unbounded directions of $\widetilde{G}_p$ in Lemma 3.4.

It can be easily verified that $\widetilde{L}_\ell$ is a closed cone $(\ell = 0, 1, 2, \ldots, p)$. Thus, we have

$$\text{conv}\ \widetilde{L}_\ell = \left\{\sum_{j=1}^{q} \boldsymbol{d}_j\boldsymbol{d}_j^T : \boldsymbol{d}_j\boldsymbol{d}_j^T \in \widetilde{L}_\ell\ (j = 1, 2, \ldots, q) \text{ for some } q \geq 0\right\}.$$

In the remainder of this section, we establish

$$\text{cl conv } \widetilde{G}_p = \text{conv } \widetilde{G}_p + \text{conv } \widetilde{L}_p = \widehat{G}_p \tag{3.6}$$

with additional assumptions. Note that $\widetilde{L}_\ell = \{\boldsymbol{O}\}$ ($\ell = 0, 1, \ldots, p$) if Condition (A) holds. In this case, we have already confirmed that conv $\widetilde{G}_p = \widehat{G}_p$ in Section 3.1.

LEMMA 3.3.
(i) *cl conv* $\widetilde{G}_\ell \subset$ *conv* $\widetilde{G}_\ell +$ *conv* $\widetilde{L}_\ell$ ($\ell = 0, 1, \ldots, p$).
(ii) *conv* $\widetilde{G}_\ell +$ *conv* $\widetilde{L}_\ell \subset \widehat{G}_\ell$ ($\ell = 0, 1, \ldots, p$).

Proof. See Section 4.1.

We now introduce additional conditions.

(A)' $\mathbb{R} \ni \rho > 0$ and $\boldsymbol{O} \neq \boldsymbol{H}_0 \in \mathbb{C}_{\mathbb{K}}$

($\widetilde{\text{C}}$) For every $\ell = 1, 2, \ldots, p$,

$$\boldsymbol{H}_\ell \bullet \boldsymbol{d}\boldsymbol{d}^T \geq 0 \text{ if } \boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_{\ell-1}. \tag{3.7}$$

(D) Every nonzero $\boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_p$ is an asymptotic unbounded direction of $\widetilde{G}_p$.

Condition (A)' is weaker than Condition (A). Specifically, we can now choose any nonzero $\boldsymbol{H}_0 \in \mathbb{C}_{\mathbb{K}}$ to satisfy Condition (A)', and $\widetilde{G}_p$ can be unbounded. We also mention that Condition (D) involves only p, but not ℓ for $\ell = 0, 1, \ldots, p$. If

(D)' every nonzero $\boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_\ell$ is an asymptotic unbounded direction of $\widetilde{G}_\ell$

holds for some $\ell = 0, 1, \ldots, p$, we can prove that $\widetilde{L}_\ell = \{\boldsymbol{O}\}$ if and only if $\widetilde{G}_\ell$ is bounded.

Let $\ell \in \{1, 2, \ldots, p\}$ be fixed. It can be easily verified that the following three statements are equivalent:

$$\text{Both (3.3) and (3.7) hold.} \tag{3.8}$$

$$\boldsymbol{H}_\ell \bullet \boldsymbol{X} \geq 0 \text{ if } \boldsymbol{X} \in \text{conv } \widetilde{G}_{\ell-1} + \text{conv } \widetilde{L}_{\ell-1}. \tag{3.9}$$

$$\boldsymbol{H}_\ell \bullet \boldsymbol{X} \geq 0 \text{ if } \boldsymbol{X} \in \text{cone conv } \widetilde{G}_{\ell-1} + \text{conv } \widetilde{L}_{\ell-1}. \tag{3.10}$$

Using almost the same argument as in the proof of (iii) of Lemma 3.4 (see the first paragraph of Section 4.2), we can prove that cone conv $\widetilde{G}_0 +$ conv $\widetilde{L}_0 = \mathbb{C}^*_{\mathbb{K}}$. Hence, we need to choose $\boldsymbol{H}_1$ from $\mathbb{C}_{\mathbb{K}}$ under Conditions ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$).

Suppose that $\ell \geq 2$. Obviously, we have

$$\text{cone conv } \widetilde{G}_{\ell-1} + \text{conv } \widetilde{L}_{\ell-1} \supset \text{cone conv } \widetilde{G}_\ell + \text{conv } \widetilde{L}_\ell. \tag{3.11}$$

If this inclusion is not proper, *i.e.* cone conv $\widetilde{G}_{\ell-1} +$ conv $\widetilde{L}_{\ell-1} =$ cone conv $\widetilde{G}_\ell +$ conv $\widetilde{L}_\ell$, then (3.10) implies that

$$\boldsymbol{H}_\ell \bullet \boldsymbol{X} \geq 0 \text{ and } \boldsymbol{H}_{\ell+1} \bullet \boldsymbol{X} \geq 0 \;\; \text{for every } \boldsymbol{X} \in \text{cone conv } \widetilde{G}_{\ell-1} + \text{conv } \widetilde{L}_{\ell-1}.$$

In this case, we have

$$\begin{aligned}
\widetilde{G}_{\ell+1} &= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_\ell : \boldsymbol{H}_{\ell+1} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\} \\
&= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1} : \boldsymbol{H}_\ell \bullet \boldsymbol{x}\boldsymbol{x}^T = 0,\; \boldsymbol{H}_{\ell+1} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\} \\
&= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1} : (\boldsymbol{H}_\ell + \boldsymbol{H}_{\ell+1}) \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\}, \\
\widetilde{L}_{\ell+1} &= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{L}_{\ell-1} : (\boldsymbol{H}_\ell + \boldsymbol{H}_{\ell+1}) \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\}.
\end{aligned}$$

Thus, we can reconstruct the hierarchical structure such that

$$\begin{aligned}\widetilde{G}_\ell &= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_{\ell-1} : (\boldsymbol{H}_\ell + \boldsymbol{H}_{\ell+1}) \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\},\\ \widetilde{G}_{\ell+2} &= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_\ell : \boldsymbol{H}_{\ell+2} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\},\\ \widetilde{L}_\ell &= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{L}_{\ell-1} : (\boldsymbol{H}_\ell + \boldsymbol{H}_{\ell+1}) \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\},\\ \widetilde{L}_{\ell+2} &= \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{L}_\ell : \boldsymbol{H}_{\ell+2} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0\right\}\end{aligned}$$

and skip the hierarchical level $\ell+1$. Consequently, we may assume under Conditions ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$) that the inclusion in (3.11) is proper. This implies that the dual cone of cone conv $\widetilde{G}_{\ell-1}$ + conv $\widetilde{L}_{\ell-1}$, which $\boldsymbol{H}_\ell$ needs to be chosen from due to (3.10), expands monotonically as ℓ increases from 2 through p.

We now consider a simple example to show why Condition (D) is necessary. Let

$$n = 2,\ p = 1,\ \rho = 1,\ \mathbb{K} = \mathbb{R}^2_+,\ \boldsymbol{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^2_+,\ \boldsymbol{H}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{C},$$

to define $\widetilde{G}_0$ and $\widetilde{G}_1$ by (3.1), and $\widetilde{L}_0$ and $\widetilde{L}_1$ by (3.5). Then,

$$\begin{aligned}\widetilde{G}_1 &= \left\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2_+,\ x_1 = 1,\ x_1x_2 = 0\right\} = \{(1,0)\},\\ \widetilde{L}_1 &= \left\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2_+,\ x_1 = 0,\ x_1x_2 = 0\right\} = \{(0,x_2) : x_2 \geq 0\}.\end{aligned}$$

Thus, cl conv $\widetilde{G}_1 = \{(1,0)\} \neq \{(1,x_2) : x_2 \geq 0\} =$ conv $\widetilde{G}_1$ + conv $\widetilde{L}_1$. Notice that it has resulted in cl conv $\widetilde{G}_1 \neq$ conv $\widetilde{G}_1$ + conv $\widetilde{L}_1$, even with a closed bounded convex set $\widetilde{G}_1$. This simple example does not satisfy Condition (D) because $\widetilde{L}_1$ contains a nonzero $\boldsymbol{d}\boldsymbol{d}^T$ with $\boldsymbol{d} = (0,1)$ and $\widetilde{G}_1$ is bounded.

LEMMA 3.4.

(iii) Assume Conditions (A)', ($\widetilde{B}$) and ($\widetilde{C}$). Then, conv $\widetilde{G}_\ell$ + *conv* $\widetilde{L}_\ell \supset \widehat{G}_\ell$ $(\ell = 0, 1, \ldots, p)$.

(iv) Assume Condition (D). Then, cl conv $\widetilde{G}_p \supset$ *conv* $\widetilde{G}_p$ + *conv* $\widetilde{L}_p$.

Proof. See Section 4.2.

The assumptions and the conclusion of (iii) of Lemma 3.4 imply that (3.8), (3.9) and (3.10) with $\ell = 1, 2, \ldots, p$ as well as Condition ($\widehat{B}$) hold. On the other hand, if Condition ($\widehat{B}$) is satisfied then either of (3.8), (3.9) and (3.10) with $\ell = 1, 2, \ldots, p$ holds by (ii) of Lemma 3.3; hence Conditions ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$) follow. Therefore, the two conditions ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$) can be combined into ($\widehat{\text{B}}$) without weakening the assertion (iii) of Lemma 3.4. By Lemmas 3.3 and 3.4, we obtain:

THEOREM 3.5. *Assume Conditions (A)', ($\widehat{B}$) and (D) (or equivalently, Conditions (A)', ($\widetilde{B}$), ($\widetilde{C}$) and (D)). Then, the identity (3.6) holds. Moreover, if* $\widetilde{G}_p$ *is bounded (hence* $\widetilde{L}_p = \{\boldsymbol{O}\}$ *by Condition (D)) then conv* $\widetilde{G}_p = \widehat{G}_p$.

4. Proofs. Before presenting the proofs of Lemmas 3.3 and 3.4, we describe a characterization of points in $\widetilde{G}_\ell$ and conv $\widetilde{L}_\ell$. Let $\ell \in \{0, 1, \ldots, p\}$. We know that $\boldsymbol{Y} \in$ conv $\widetilde{G}_\ell$ if and only if there exist $\boldsymbol{Y}_i \in \widetilde{G}_\ell$, $\boldsymbol{y}_i \in \mathbb{K}$ and $\lambda_i \in \mathbb{R}$ $(i = 1, 2, \ldots, r)$

such that

$$(4.1)\quad \begin{array}{rcl} \boldsymbol{Y} & = & \sum_{i=1}^{r} \lambda_i \boldsymbol{Y}_i,\ \sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i \geq 0 \ \ (i = 1, 2, \dots, r), \\ & & \boldsymbol{Y}_i \in \widetilde{G}_\ell \ (i = 1, 2, \dots, r), \textit{i.e.,} \\ \boldsymbol{Y}_i & = & \boldsymbol{y}_i(\boldsymbol{y}_i)^T, \boldsymbol{y}_i \in \mathbb{K},\ \boldsymbol{H}_0 \bullet \boldsymbol{Y}_i = \rho, \\ & & \boldsymbol{H}_k \bullet \boldsymbol{Y}_i = 0 \ (k = 1, 2, \dots, \ell) \ (i = 1, 2, \dots, r). \end{array}$$

Note that $\boldsymbol{D} \in \mathrm{conv}\ \widetilde{L}_\ell$ if an only if there exist $\boldsymbol{d}_j \in \mathbb{K}$ $(j = 1, 2, \dots, q)$ such that

$$(4.2)\qquad \boldsymbol{D} = \sum_{j=1}^{q} \boldsymbol{d}_j \boldsymbol{d}_j^T,\ \boldsymbol{H}_k \bullet \boldsymbol{d}_j \boldsymbol{d}_j^T = 0 \ (k = 0, 1, \dots, \ell) \ (j = 1, 2, \dots, q).$$

We can fix both r and q so that $\dim \mathbb{S}^n + 1 = n(n+1)/2 + 1$ (Carathéodory's theorem).

4.1. Proof of Lemma 3.3. Proof of (i) cl conv $\widetilde{G}_\ell \subset$ conv $\widetilde{G}_\ell +$ conv $\widetilde{L}_\ell$ $(\ell = 0, 1, \dots, p)$: Let $\ell \in \{0, 1, \dots, p\}$. Assume that $\boldsymbol{X} \in \mathrm{cl\ conv}\ \widetilde{G}_\ell$. Then, there is a sequence $\{\boldsymbol{X}^s \in \mathrm{conv}\ \widetilde{G}_\ell : s = 1, 2, \dots\}$ converging to $\boldsymbol{X}$. Each $\boldsymbol{Y} = \boldsymbol{X}^s$ is characterized by (4.1) for some $\boldsymbol{Y}_i = \boldsymbol{X}_i^s \in \widetilde{G}_\ell$, $\boldsymbol{y}_i = \boldsymbol{x}_i^s \in \mathbb{K}$ and $\lambda_i = \lambda_i^s \in \mathbb{R}$ $(i = 1, 2, \dots, r)$. Since both $\sqrt{\lambda_i^s}\boldsymbol{x}_i^s(\sqrt{\lambda_i^s}\boldsymbol{x}_i^s)^T$ and $\boldsymbol{X}^s - \sqrt{\lambda_i^s}\boldsymbol{x}_i^s(\sqrt{\lambda_i^s}\boldsymbol{x}_i^s)^T$ are positive semidefinite $(i = 1, 2, \dots, r, s = 1, 2, \dots)$ and $\boldsymbol{X}^s \to \boldsymbol{X}$ as $s \to \infty$, the sequence

$$\left\{ \left(\sqrt{\lambda_1^s}\boldsymbol{x}_1^s, \sqrt{\lambda_2^s}\boldsymbol{x}_2^s, \dots, \sqrt{\lambda_r^s}\boldsymbol{x}_r^s \right) : s = 1, 2, \dots \right\}$$

is bounded. And, the sequence $\{(\lambda_1^s, \lambda_2^s, \dots, \lambda_r^s) : s = 1, 2, \dots\}$ is also bounded. We may assume without loss of generality that

$$\begin{array}{r} \left(\sqrt{\lambda_1^s}\boldsymbol{x}_1^s, \sqrt{\lambda_2^s}\boldsymbol{x}_2^s, \dots, \sqrt{\lambda_r^s}\boldsymbol{x}_r^s \right) \to (\boldsymbol{d}_1, \boldsymbol{d}_2, \dots, \boldsymbol{d}_r) \ \text{ and} \\ (\lambda_1^s, \lambda_2^s, \dots, \lambda_r^s) \to (\lambda_1, \lambda_2, \dots, \lambda_r) \end{array}$$

as $s \to \infty$ for some $(\boldsymbol{d}_1, \boldsymbol{d}_2, \dots, \boldsymbol{d}_r)$ and $(\lambda_1, \lambda_2, \dots, \lambda_r)$. Let

$$I_{\mathrm{bd}} = \left\{ i : \sup_s \|\boldsymbol{x}_i^s\| < \infty \right\} \ \text{ and } I_\infty = \left\{ j : \sup_s \|\boldsymbol{x}_j^s\| = \infty \right\}.$$

Then, we can take a subsequence of $\{(\boldsymbol{x}_1^s, \boldsymbol{x}_2^s, \dots, \boldsymbol{x}_r^s)\}$ along which

$$\boldsymbol{x}_i^s \to \boldsymbol{x}_i \text{ for some } \boldsymbol{x}_i \in \mathbb{K} \ (i \in I_{\mathrm{bd}}) \ \text{ and } \|\boldsymbol{x}_j^s\| \to \infty,\ \lambda_j^s \to 0 \ (j \in I_\infty).$$

Consequently,

$$\begin{aligned} \boldsymbol{X}^s &= \sum_{i \in I_{\mathrm{bd}}} \lambda_i^s \boldsymbol{x}_i^s (\boldsymbol{x}_i^s)^T + \sum_{j \in I_\infty} (\sqrt{\lambda_j^s}\boldsymbol{x}_j^s)(\sqrt{\lambda_j^s}\boldsymbol{x}_j^s)^T, \\ 1 &= \sum_{i \in I_{\mathrm{bd}}} \lambda_i^s + \sum_{j \in I_\infty} \lambda_j^s,\ \lambda_i^s \geq 0 \ (i \in I_{\mathrm{bd}}),\ \boldsymbol{x}_i^s \in \mathbb{K} \ (i \in I_{\mathrm{bd}}), \\ & \quad \sqrt{\lambda_j^s}\boldsymbol{x}_j^s \in \mathbb{K} \ (j \in I_\infty), \\ \rho &= \boldsymbol{H}_0 \bullet \boldsymbol{X}_i^s = \boldsymbol{H}_0 \bullet \boldsymbol{x}_i^s (\boldsymbol{x}_i^s)^T \ \ (i \in I_{\mathrm{bd}}), \\ \lambda_i^s \rho &= \lambda_i^s \left(\boldsymbol{H}_0 \bullet \boldsymbol{x}_i^s (\boldsymbol{x}_i^s)^T \right) = \boldsymbol{H}_0 \bullet \sqrt{\lambda_i^s}\boldsymbol{x}_i^s \left(\sqrt{\lambda_i^s}\boldsymbol{x}_i^s \right)^T \ \ (i \in I_\infty), \\ 0 &= \boldsymbol{H}_k \bullet \boldsymbol{X}_i^s = \boldsymbol{H}_k \bullet \boldsymbol{x}_i^s (\boldsymbol{x}_i^s)^T \ (k = 1, 2, \dots, \ell) \ (i \in I_{\mathrm{bd}}), \\ 0 &= \lambda_i^s \times \boldsymbol{H}_k \bullet \boldsymbol{X}_i^s = \boldsymbol{H}_k \bullet \sqrt{\lambda_i^s}\boldsymbol{x}_i^s \left(\sqrt{\lambda_i^s}\boldsymbol{x}_i^s \right)^T \ \ (k = 1, 2, \dots, \ell) \ (i \in I_\infty). \end{aligned}$$

Taking the limit along the subsequence, we obtain

$$
\begin{aligned}
\boldsymbol{X} &= \sum_{i\in I_{\rm bd}} \lambda_i \boldsymbol{x}_i(\boldsymbol{x}_i)^T + \sum_{j\in I_\infty} \boldsymbol{d}_j\boldsymbol{d}_j^T, \\
1 &= \sum_{i\in I_{\rm bd}} \lambda_i,\ \lambda_i \geq 0\ (i\in I_{\rm bd}),\ \boldsymbol{x}_i\in\mathbb{K}\ (i\in I_{\rm bd}),\ \boldsymbol{d}_j\in\mathbb{K}\ (j\in I_\infty), \\
\rho &= \boldsymbol{H}_0\bullet\boldsymbol{x}_i\boldsymbol{x}_i^T\ (i\in I_{\rm bd}),\ 0=\boldsymbol{H}_0\bullet\boldsymbol{d}_j\boldsymbol{d}_j^T\ (j\in I_\infty), \\
0 &= \boldsymbol{H}_k\bullet\boldsymbol{x}_i\boldsymbol{x}_i^T\ (i\in I_{\rm bd}),\ 0=\boldsymbol{H}_k\bullet\boldsymbol{d}_j\boldsymbol{d}_j^T\ (j\in I_\infty),\ (k=1,2,\ldots,\ell).
\end{aligned}
\tag{4.3}
$$

Thus, we have shown that $\boldsymbol{X}\in$ conv $\widetilde{G}_\ell+$conv $\widetilde{L}_\ell$.

∎

Proof of (ii) conv $\widetilde{G}_\ell+$ conv $\widetilde{L}_\ell\subset\widehat{G}_\ell$ $(\ell=0,1,\ldots,p)$: Let $\ell\in\{0,1,\ldots,p\}$. Suppose that $\boldsymbol{X}\in$ conv $\widetilde{G}_\ell+$conv $\widetilde{L}_\ell$. Then, $\boldsymbol{X}=\boldsymbol{Y}+\boldsymbol{D}$ for some $\boldsymbol{Y}\in$ conv $\widetilde{G}_\ell$ and some $\boldsymbol{D}\in$ conv $\widetilde{L}_\ell$. Recall that $\boldsymbol{Y}\in$ conv $\widetilde{G}_\ell$ and $\boldsymbol{D}\in$ conv $\widetilde{L}_\ell$ are characterized by (4.1) and (4.2), respectively. Hence,

$$
\boldsymbol{X}=\sum_{i=1}^r(\sqrt{\lambda_i}\boldsymbol{y}_i)(\sqrt{\lambda_i}\boldsymbol{y}_i)^T+\sum_{j=1}^q\boldsymbol{d}_j\boldsymbol{d}_j\in\mathbb{C}^*_{\mathbb{K}},
$$

$$
\boldsymbol{H}_0\bullet\boldsymbol{X}=\boldsymbol{H}_0\bullet\left(\sum_{i=1}^r\lambda_i\boldsymbol{Y}_i+\sum_{j=1}^q\boldsymbol{d}_j\boldsymbol{d}_j\right)=\sum_{i=1}^r\lambda_i\left(\boldsymbol{H}_0\bullet\boldsymbol{Y}_i\right)=\rho\sum_{i=1}^r\lambda_i=\rho,
$$

$$
\boldsymbol{H}_k\bullet\boldsymbol{X}=\boldsymbol{H}_k\bullet\left(\sum_{i=1}^r\lambda_i\boldsymbol{Y}_i+\sum_{j=1}^q\boldsymbol{d}_j\boldsymbol{d}_j\right)=0\ (k=1,2,\ldots,\ell).
$$

This implies $\boldsymbol{X}\in\widehat{G}_\ell$. ∎

4.2. Proof of Lemma 3.4. Proof of (iii) conv $\widetilde{G}_\ell+$ conv $\widetilde{L}_\ell\supset\widehat{G}_\ell$ $(\ell=0,1,\ldots,p)$ under Condition (A)', $(\widetilde{\rm B})$ and $(\widetilde{\rm C})$: We use an induction argument to prove conv $\widetilde{G}_\ell+$ conv $\widetilde{L}_\ell\supset\widehat{G}_\ell$ $(\ell=0,1,\ldots,)$. Let $\ell=0$. Assume that $\boldsymbol{X}\in\widehat{G}_0$. Then, there exist $\boldsymbol{x}_i\in\mathbb{K}$ $(i=1,2,\ldots,r)$ such that

$$
\boldsymbol{X}=\sum_{i=1}^r\boldsymbol{x}_i\boldsymbol{x}_i^T,\ \boldsymbol{H}_0\bullet\boldsymbol{X}=\rho.
$$

Let $\lambda_i=(\boldsymbol{H}_0\bullet\boldsymbol{x}_i\boldsymbol{x}_i)/\rho$ $(i=1,2,\ldots,r)$. By Condition (A)', $\lambda_i\geq 0$ $(i=1,2,\ldots,r)$. Let

$$
I_+=\{i:\lambda_i>0\},\ I_0=\{i:\lambda_i=0\},\ \boldsymbol{y}_i=\boldsymbol{x}_i/\sqrt{\lambda_i}\in\mathbb{K}\ (i\in I_+).
$$

Then,

$$
\boldsymbol{X}=\sum_{i\in I_+}\lambda_i\boldsymbol{y}_i\boldsymbol{y}_i^T+\sum_{j\in I_0}\boldsymbol{x}_j\boldsymbol{x}_j^T,\ \lambda_i>0\ (i\in\boldsymbol{I}_+),
$$

$$
\sum_{i\in I_+}\lambda_i=\sum_{i\in I_+}(\boldsymbol{H}_0\bullet\boldsymbol{x}_i\boldsymbol{x}_i)/\rho=\sum_{i=1}^r(\boldsymbol{H}_0\bullet\boldsymbol{x}_i\boldsymbol{x}_i)/\rho=(\boldsymbol{H}_0\bullet\boldsymbol{X})/\rho=1,
$$

$$
\begin{aligned}
&\boldsymbol{H}_0\bullet\boldsymbol{y}_i\boldsymbol{y}_i^T=\boldsymbol{H}_0\bullet(\boldsymbol{x}_i/\sqrt{\lambda_i})(\boldsymbol{x}_i/\sqrt{\lambda_i})^T=(\boldsymbol{H}_0\bullet\boldsymbol{x}_i\boldsymbol{x}_i^T)/\lambda_i=\rho\ (i\in I_+),\\
&\boldsymbol{H}_0\bullet\boldsymbol{x}_j\boldsymbol{x}_j^T=0\ (j\in I_0).
\end{aligned}
$$

This implies $\boldsymbol{X} \in \text{conv } \widetilde{G}_0 + \text{conv } \widetilde{L}_0$.

Now, we assume that $\widehat{G}_k \subset \text{conv } \widetilde{G}_k + \text{conv } \widetilde{L}_k$ $(k = 0, 1, \ldots, \ell - 1)$ holds with $1 \leq \ell \leq p$ and prove $\widehat{G}_\ell \subset \text{conv } \widetilde{G}_\ell + \text{conv } \widetilde{L}_\ell$. Suppose that $\boldsymbol{X} \in \widehat{G}_\ell$. Since $\widehat{G}_\ell \subset \widehat{G}_{\ell-1} \subset \text{conv } \widetilde{G}_{\ell-1} + \text{conv } \widetilde{L}_{\ell-1}$, we see that $\boldsymbol{X} \in \text{conv } \widetilde{G}_{\ell-1} + \text{conv } \widetilde{L}_{\ell-1}$. Thus,

$$\begin{aligned}
&\boldsymbol{X} = \sum_{i=1}^{r} \lambda_i \boldsymbol{Y}_i + \sum_{j=1}^{q} \boldsymbol{d}_j \boldsymbol{d}_j^T, \ \lambda_i > 1 \ (i = 1, 2, \ldots, r), \ \sum_{i=1}^{r} \lambda_i = 1, \\
&\boldsymbol{Y}_i \in \widetilde{G}_{\ell-1} \ (i = 1, 2, \ldots, r), \ \boldsymbol{d}_j \boldsymbol{d}_j^T \in \ \widetilde{L}_{\ell-1} \ (j = 1, 2, \ldots, q).
\end{aligned}$$

To complete the proof, it suffices to show that $\boldsymbol{Y}_i \in \widetilde{G}_\ell$ $(i = 1, 2, \ldots, r)$ and $\boldsymbol{d}_j \boldsymbol{d}_j^T \in \ \widetilde{L}_\ell$ $(j = 1, 2, \ldots, q)$. By Conditions $(\widetilde{\text{B}})$ and $(\widetilde{\text{C}})$, we have

$$\boldsymbol{H}_\ell \bullet \boldsymbol{Y}_i \geq 0 \ (i = 1, 2, \ldots, r) \ \text{ and } \boldsymbol{H}_\ell \bullet \boldsymbol{d}_j \boldsymbol{d}_j^T \geq 0 \ (j = 1, 2, \ldots, q).$$

On the other hand, it follows from $\boldsymbol{X} \in \widehat{G}_\ell$ that

$$0 = \boldsymbol{H}_\ell \bullet \boldsymbol{X} = \sum_{i=1}^{r} \lambda_i (\boldsymbol{H}_\ell \bullet \boldsymbol{Y}_i) + \sum_{j=1}^{q} \boldsymbol{H}_\ell \bullet \boldsymbol{d}_j \boldsymbol{d}_j^T.$$

Since $\lambda_i > 0$ $(i = 1, 2, \ldots, r)$, we obtain that

$$\boldsymbol{H}_\ell \bullet \boldsymbol{Y}_i = 0 \ (i = 1, 2, \ldots, r), \ \boldsymbol{H}_\ell \bullet \boldsymbol{d}_j \boldsymbol{d}_j^T = 0 \ (j = 1, 2, \ldots, q).$$

Thus, we have shown that $\boldsymbol{Y}_i \in \widetilde{G}_\ell$ $(i = 1, 2, \ldots, r)$ and $\boldsymbol{d}_j \boldsymbol{d}_j^T \in \ \widetilde{L}_\ell$ $(j = 1, 2, \ldots, q)$. ∎

Proof of (iv) cl conv $\widetilde{G}_p \supset$ conv $\widetilde{G}_p +$ conv $\widetilde{L}_p$ under Condition (D): Suppose that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{D}$ for some $\boldsymbol{Y} \in \text{conv } \widetilde{G}_p$ and $\boldsymbol{D} \in \text{conv } \widetilde{L}_p$. Then, we have (4.1) with $\ell = p$ for some $\boldsymbol{Y}_i \in \widetilde{G}_\ell$, $\boldsymbol{y}_i \in \mathbb{K}$ and $\lambda_i \in \mathbb{R}$ $(i = 1, 2, \ldots, r)$, and (4.2) with $\ell = p$ for some $\boldsymbol{d}_j \in \mathbb{K}$ $(j = 1, 2, \ldots, q)$. By Condition (D), for every $j = 1, 2, \ldots, q$, there is a sequence

$$\left\{ (\mu_j^s, \boldsymbol{u}_j^s (\boldsymbol{u}_j^s)^T) \in \mathbb{R}_+ \times \widetilde{G}_p : s = 1, 2, \ldots \right\}$$

such that $\|\boldsymbol{u}_j^s\| \to \infty$ and $(\sqrt{\mu_j^s}, \sqrt{\mu_j^s} \boldsymbol{u}_j^s) \to (0, \boldsymbol{d}_j)$ as $\mu \to \infty$. For every $s = 1, 2 \ldots$, let

$$\begin{aligned}
&\gamma^s = 1 + \sum_{j=1}^{q} \mu_j^s, \ \lambda_i^s = \lambda_i / \gamma^s \ (i = 1, 2, \ldots, r), \ \nu_j^s = \mu_j^s / \gamma^s \ (j = 1, 2, \ldots, q), \\
&\boldsymbol{X}^s = \sum_{i=1}^{r} \lambda_i^s \boldsymbol{y}_i \boldsymbol{y}_i^T + \sum_{j=1}^{q} \nu_j^s \boldsymbol{u}_j^s (\boldsymbol{u}_j^s)^T.
\end{aligned}$$

Then,

$$\begin{aligned}
&\lambda_i^s \geq 0, \ \nu_j^s \geq 0, \ \sum_{i=1}^{r} \lambda_i^s + \sum_{j=1}^{q} \nu_j^s = 1, \\
&\boldsymbol{y}_i \boldsymbol{y}_i^T \in \widetilde{G}_p \ (i = 1, 2, \ldots, r), \ \boldsymbol{u}_j^s (\boldsymbol{u}_j^s)^T \in \widetilde{G}_p \ (j = 1, 2, \ldots, q),
\end{aligned}$$

as a result, $\boldsymbol{X}^s \in \text{conv}\ \widetilde{G}_p$ $(s = 1, 2, \dots)$. We can also verify that

$$\gamma^s \to 1,\ \lambda_i^s \to \lambda_i\ (i = 1, 2, \dots. r),\ \nu_j^s \to 0\ (j = 1, 2, \dots, q),$$
$$\nu_j^s \boldsymbol{u}_j^s (\boldsymbol{u}_j^s)^T \to \boldsymbol{d}_j \boldsymbol{d}_j^T,\ \text{conv}\ \widetilde{G}_p \ni \boldsymbol{X}^s \to \boldsymbol{X}$$

as $s \to \infty$. Thus, we have shown that $\boldsymbol{X} \in \text{cl conv}\ \widetilde{G}_p$. ∎

5. Examples. We present four examples to show the QOP model (2.1) covers various types of nonconvex QOPs. The first example is a QOP with linear equality constraints in nonnegative continuous variables and binary variables, and an additional constraint $\boldsymbol{u} \in D$ in its variable vector $\boldsymbol{u}$, where D is a closed subset of $\mathbb{R}^m$. This type of problems was studied in [11] as an extension of a QOP with linear equality constraints in nonnegative continuous variables and binary variables studied in [5]. The second example shows how the hierarchy of constraint set $\widetilde{G}_\ell$ $(\ell = 0, 1, \dots, p)$ satisfying Conditions (A) and ($\widehat{\text{B}}$) can be constructed for complicated combinatorial constraints. The last two examples demonstrate that QOP (2.1) satisfying Conditions (A)', ($\widehat{\text{B}}$) and (D) can deal with new types of nonconvex QOPs, although they may look somewhat unnatural.

5.1. A QOP with linear equality constraints in nonnegative continuous variables and binary variables, and an additional constraint $\boldsymbol{u} \in D$ in its variable vector $\boldsymbol{u}$. Let D be a closed subset of $\mathbb{R}^m$, $\boldsymbol{A}$ a $q \times m$ matrix, $\boldsymbol{b} \in \mathbb{R}^q$ and $r \leq m$ a positive integer. We consider a QOP of the form

$$\begin{array}{lll} & \text{minimize} & \boldsymbol{u}^T \boldsymbol{Q}_0 \boldsymbol{u} + 2\boldsymbol{c}^T \boldsymbol{u} \\ (5.1) & \text{subject to} & \boldsymbol{u} \in D,\ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} = \boldsymbol{0},\ u_i(1 - u_i) = 0\ (i = 1, 2, \dots, r). \end{array}$$

If $D = \mathbb{R}^m_+$ or $\boldsymbol{u} \geq \boldsymbol{0}$, this QOP model coincides with the one studied in [6]. Define $\mathbb{K} = \text{cl}\ \left\{(u_0, u_0\boldsymbol{u}) \in \mathbb{R}^{1+m}_+ : u_0 \in \mathbb{R}_+,\ \boldsymbol{u} \in D\right\}$. Then, we can rewrite QOP (5.1) as

$$\begin{array}{lll} & \text{minimize} & \boldsymbol{u}^T \boldsymbol{Q}_0 \boldsymbol{u} + 2\boldsymbol{c}^T u_0 \boldsymbol{u} \\ (5.2) & \text{subject to} & (u_0, \boldsymbol{u}) \in \mathbb{K},\ u_0 = 1,\ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}u_0 = \boldsymbol{0}, \\ & & u_i(u_0 - u_i) = 0\ (i = 1, 2, \dots, r). \end{array}$$

We assume that

$$(5.3) \qquad 0 \leq u_i \leq u_0\ (i = 1, 2, \dots, r)\ \text{ if } (u_0, \boldsymbol{u}) \in \mathbb{K},\ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}u_0 = \boldsymbol{0}.$$

Note that $0 \leq u_i \leq u_0$ implies $u_i(u_0 - u_i) \geq 0$ $(1, 2, \dots, r)$. Thus, we can replace the multiple quadratic equalities $u_i(u_0 - u_i) = 0$ $(i = 1, 2, \dots, r)$ in QOP (5.2) by a single equality $\sum_{i=1}^r u_i(u_0 - u_i) = 0$, and we see that

$$(5.4) \qquad \sum_{i=1}^r u_i(u_0 - u_i) \geq 0 \text{ if } (u_0, \boldsymbol{u}) \in \mathbb{K},\ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}u_0 = \boldsymbol{0}.$$

Let $\rho = 1$. We rewrite the problem as

$$\begin{array}{ll} \text{minimize} & \boldsymbol{u}^T \boldsymbol{Q}_0 \boldsymbol{u} + 2u_0 \boldsymbol{c}^T \boldsymbol{u} \\ \text{subject to} & (u_0, \boldsymbol{u}) \in \mathbb{K},\ u_0^2 = \rho, (A\boldsymbol{u} - \boldsymbol{b}u_0)^T(A\boldsymbol{u} - \boldsymbol{b}u_0) = 0, \\ & \displaystyle\sum_{i=1}^r u_i(u_0 - u_i) = 0. \end{array}$$

Let $n = 1 + m$. To represent the quadratic form of the problem above in the form of QOP (2.1), we introduce a variable vector $\boldsymbol{x} = (u_0, \boldsymbol{u}) \in \mathbb{R}^n$ and take matrices $\boldsymbol{Q} \in \mathbb{S}^n$, $\boldsymbol{H}_k \in \mathbb{S}^n$ $(k = 0, 1, 2)$ such that the following identities hold.

$$\boldsymbol{Q} \bullet \boldsymbol{x}\boldsymbol{x}^T = \boldsymbol{u}^T \boldsymbol{Q}_0 \boldsymbol{u} + 2u_0 \boldsymbol{c}^T \boldsymbol{u}, \ \boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T = u_0^2,$$
$$\boldsymbol{H}_1 \bullet \boldsymbol{x}\boldsymbol{x}^T = (\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}u_0)^T(\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}u_0), \ \boldsymbol{H}_2 \bullet \boldsymbol{x}\boldsymbol{x}^T = \sum_{i=1}^{r} u_i(u_0 - u_i).$$

Defining $\widetilde{G}_0, \ \widetilde{G}_1, \ \widetilde{G}_2$ as in (3.1), we can finally rewrite the problem as a QOP of the form (2.1) with $p = 2$. It is trivial to confirm that Condition (A)' is satisfied, and Conditions ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$) are satisfied by $\boldsymbol{H}_0 \in \mathbb{S}^n_+$, $\boldsymbol{H}_1 \in \mathbb{S}^n_+$ and (5.4).

For Condition (D), we need an additional assumption on D.

(E) If $\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^m$ is an asymptotic unbounded direction of D, *i.e.*, there is a sequence

$$\{(\mu^s, \boldsymbol{u}^s) \in \mathbb{R}_+ \times D : s = 1, 2, \dots \}$$

such that $\|\boldsymbol{u}^s\| \to \infty$ and $(\mu^s, \mu^s \boldsymbol{u}^s) \to (0, \boldsymbol{v})$ as $s \to \infty$, then, for every $\boldsymbol{u} \in D$, there exists a sequence $\{\nu^s : s = 1, 2, \dots \}$ of positive numbers such that

$$\boldsymbol{u} + \nu^s \boldsymbol{v} \in D \ (s = 1, 2, \dots) \ \text{ and } \nu^s \to \infty \ \text{ as } s \to \infty. \tag{5.5}$$

By definition, D satisfies (E) if it is bounded. We can prove that if D is convex, then it satisfies (E); more precisely every asymptotic unbounded direction $\boldsymbol{v}$ of D is an unbounded direction such that $\boldsymbol{u} + \nu \boldsymbol{v} \in D$ for every $\nu \geq 0$ and $\boldsymbol{u} \in D$. For other examples, the set $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 = \sin x_1\right\}$ is a nonconvex set that satisfies (E). A typical example that does not satisfy (E) is the set of points characterized by complementarity $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 x_2 = 0\}$. Also the set $\left\{(x_1, x_2) \in \mathbb{R}^2_+ : 1 \leq x_1, \ x_1^2 - x_2^2 \leq 1\right\}$ does not satisfy (E).

Now, assume that (E) holds. Let $\boldsymbol{O} \neq \boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_2$, and choose a feasible solution $\boldsymbol{u} \in \mathbb{R}^n$ of QOP (5.1). Let $\boldsymbol{d} = (v_0, \boldsymbol{v}) \in \mathbb{R}^{1+m}$. Then, we have

$$v_0 = 0, \ (0, \boldsymbol{v}) \in \mathbb{K}, \ \boldsymbol{H}_1 \bullet \boldsymbol{d}\boldsymbol{d}^T = 0, \ \boldsymbol{H}_2 \bullet \boldsymbol{d}\boldsymbol{d}^T = \sum_{i=1}^{r} -v_i v_i = 0.$$

The identity $\boldsymbol{H}_1 \bullet \boldsymbol{d}\boldsymbol{d}^T = 0$ implies that $\boldsymbol{H}_1 \boldsymbol{d} = \boldsymbol{0}$ since $\boldsymbol{H}_1 \in \mathbb{S}^n_+$, and the last identity implies that $v_i = 0$ $(i = 1, 2, \dots, r)$ and $\boldsymbol{H}_2 \boldsymbol{d} = \boldsymbol{0}$. On the other hand, it follows from $(0, \boldsymbol{v}) \in \mathbb{K} = \text{cl } \left\{(u_0, u_0 \boldsymbol{u}) \in \mathbb{R}^n_+ : u_0 \in \mathbb{R}_+, \ \boldsymbol{u} \in D\right\}$ that $\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n$ is an asymptotic unbounded direction of D. Hence, there exists a sequence $\{\nu^s : s = 1, 2, \dots \}$ of positive numbers satisfying (5.5) with the feasible solution $\boldsymbol{u} \in \mathbb{R}^n$, which has been initially chosen. Let $u_0 = 1$ and $\boldsymbol{x} = (u_0, \boldsymbol{u}) \in \mathbb{R}^{1+n}$ be a feasible solution of QOP (5.2). Then, for every $s = 1, 2, \dots,$

$$\begin{aligned} &\boldsymbol{u} + \nu^s \boldsymbol{v} \in D; \text{hence } \boldsymbol{x} + \nu^s \boldsymbol{d} \in \mathbb{K}, \\ &\boldsymbol{H}_0 \bullet (\boldsymbol{x} + \nu^s \boldsymbol{d})(\boldsymbol{x} + \nu^s \boldsymbol{d})^T = \rho, \\ &\boldsymbol{H}_\ell \bullet (\boldsymbol{x} + \nu^s \boldsymbol{d})(\boldsymbol{x} + \nu^s \boldsymbol{d})^T = 0 \ (\ell = 1, 2). \end{aligned}$$

Hence $(\boldsymbol{x} + \nu^s \boldsymbol{d})(\boldsymbol{x} + \nu^s \boldsymbol{d}) \in \widetilde{G}_2$ $(s = 1, 2, \dots, \ )$. We also observe that $(1/\nu^s, (\boldsymbol{x} + \nu^s \boldsymbol{d})/\nu^s) \to (0, \boldsymbol{d})$ as $s \to \infty$. Thus, $\boldsymbol{d}\boldsymbol{d}^T$ is an asymptotic unbounded direction of $\widetilde{G}_2$, and (D) holds.

REMARK 5.1. Eichfelder and Povh stated in [11] an equivalence of QOP of the form (5.1) and a linear optimization problem over the dual cone of a set-semidefinite cone, which is a further generalization of the generalized completely positive cone in this paper, without any assumption on D. (D corresponds to K in [11].) Lemma 9 in [11] is essential to show the equivalence. However, there is a logical gap in its proof, so the proof is incomplete. In [9], Dickinson, Eichfelderl and Povh corrected their argument to prove the equivalence by imposing some additional assumptions on the QOP of the form (5.1). See [9, 11] for more details.

5.2. A set of complicated combinatorial constraints. Consider the set F of $\boldsymbol{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4_+$ satisfying the following conditions.

$$
\left.\begin{array}{l}
0 \leq u_i \leq 1 \ (i = 1, 2, 3, 4), \\
u_1 = 1 \text{ and/or } \ u_2 = 1, \\
u_4 = 0 \text{ or } \ u_4 = 1, \\
u_3 = 0 \text{ and/or } \ u_3 = u_1 + u_2, \\
u_4 = 0 \text{ and/or } u_1 + u_2 + u_3 = 2.
\end{array}\right\} \tag{5.6}
$$

We introduce a slack variable vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_8) \in \mathbb{R}^8_+$, and rewrite the above conditions as

$$
\begin{array}{l}
\rho = 16, \ \boldsymbol{e}^T\boldsymbol{x} = \sqrt{\rho}, \\
f_i(\boldsymbol{x}) \equiv x_i + x_{i+4} - (\boldsymbol{e}^T/\sqrt{\rho})\boldsymbol{x} = 0 \ (i = 1, 2, 3, 4), \\
g_{11}(\boldsymbol{x}) \equiv x_5 x_6 = 0, \ g_{12}(\boldsymbol{x}) \equiv x_4 x_8 = 0, \\
g_2(\boldsymbol{x}) \equiv x_3(x_1 + x_2 - x_3) = 0, \\
g_3(\boldsymbol{x}) \equiv x_4 \left((2e^T/\sqrt{\rho})\boldsymbol{x} - x_1 - x_2 - x_3\right) = 0,
\end{array}
$$

where $\boldsymbol{e} = (1, 1, \ldots, 1) \in \mathbb{R}^8$. Now let

$$
\begin{array}{l}
\mathbb{K} = \mathbb{R}^8_+, \ \widetilde{G}_0 = \left\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{K}, \ \boldsymbol{e}\boldsymbol{e}^T \bullet \boldsymbol{x}\boldsymbol{x}^T = \rho\right\}, \\
\widetilde{G}_1 = \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_0 : f_i(\boldsymbol{x}) = 0 \ (i = 1, 2, 3, 4), \ g_{11}(\boldsymbol{x}) = 0, \ g_{12}(\boldsymbol{x}) = 0\right\}, \\
\widetilde{G}_2 = \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_1 : g_2(\boldsymbol{x}) = 0\right\}, \ \widetilde{G}_3 = \left\{\boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_2 : g_3(\boldsymbol{x}) = 0\right\}.
\end{array}
$$

Then,

$$
F = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \begin{array}{l} \boldsymbol{x} = (x_1, x_2, \ldots, x_8) \in \mathbb{K}, \ \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_3 \\ \text{for some } \ (x_5, x_6, x_7, x_8) \in \mathbb{R}^4 \end{array}\right\}.
$$

We can verify that

$$
g_2(\boldsymbol{x}) \geq 0 \text{ if } \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_1 \ \text{ and } g_3(\boldsymbol{x}) \geq 0 \text{ if } \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_2. \tag{5.7}
$$

Choose a 4×8 matrix $\boldsymbol{A}$ and 8×8 symmetric matrices $\boldsymbol{H}_{11}$, $\boldsymbol{H}_k$ $(k = 0, 1, 2, 3)$ such that the following identities hold.

$$
\begin{array}{l}
\boldsymbol{H}_0 = \boldsymbol{e}\boldsymbol{e}^T \in \mathbb{S}^8_+, \ \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ f_3(\boldsymbol{x}) \\ f_4(\boldsymbol{x}) \end{pmatrix} = \boldsymbol{A}\boldsymbol{x}, \ g_{11}(\boldsymbol{x}) + g_{12}(\boldsymbol{x}) = \boldsymbol{H}_{11} \bullet \boldsymbol{x}\boldsymbol{x}^T, \\
\boldsymbol{H}_1 \equiv \boldsymbol{A}^T\boldsymbol{A} + \boldsymbol{H}_{11} \in \mathbb{S}^8_+ + \mathbb{N} \subset \mathbb{C}, \ g_2(\boldsymbol{x}) = \boldsymbol{H}_2 \bullet \boldsymbol{x}\boldsymbol{x}^T \ g_3(\boldsymbol{x}) = \boldsymbol{H}_3 \bullet \boldsymbol{x}\boldsymbol{x}^T.
\end{array} \tag{5.8}
$$

Consequently, $\widetilde{G}_\ell$ ($\ell = 0,1,2,3$) are described as in (3.1), and we confirm that Conditions (A) and ($\widetilde{\text{B}}$) hold by $\boldsymbol{H}_0 = \boldsymbol{e}\boldsymbol{e}^T \in \mathbb{S}^8_+$, $\boldsymbol{H}_1 \in \mathbb{C}$, and (5.7).

We have described the set F of $\boldsymbol{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4_+$ satisfying (5.6) in terms of our hierarchical model with three levels. This model can be reduced to the hierarchical model with one level by introducing additional slack variable vector $\boldsymbol{s} = (s_1, s_2) \in \mathbb{R}^2$. To show this, we first rewrite the 4th and 5th conditions in (5.6) as

$$\begin{aligned} &f_5(\boldsymbol{x}, s_1, \boldsymbol{s}) \equiv s_1 - (x_1 + x_2 - x_3) = 0 \ \text{ and } g_{13}(\boldsymbol{x}, \boldsymbol{s}) \equiv x_3 s_1 = 0, \\ &f_6(\boldsymbol{x}, s_1, \boldsymbol{s}) \equiv s_2 - ((2e^T/\sqrt{\rho})\boldsymbol{x} - x_1 - x_2 - x_3) = 0 \ \text{ and } g_{14}(\boldsymbol{x}, \boldsymbol{s}) \equiv x_4 s_2 = 0. \end{aligned}$$

We can add $s_1 \geq 0$ and $s_2 \geq 0$, which are implied by (5.6). Then, $x_3 \geq 0$, $s_1 \geq 0$ and $x_3 s_1 = 0$ as well as $x_4 \geq 0$, $s_2 \geq 0$ and $x_4 s_2 = 0$ form a standard complementarity condition. Now we redefine

$$\begin{aligned} &\mathbb{K} = \mathbb{R}^{10}_+, \ \widetilde{G}_0 = \left\{ (\boldsymbol{x}, \boldsymbol{s})(\boldsymbol{x}, \boldsymbol{s})^T : (\boldsymbol{x}, \boldsymbol{s}) \in \mathbb{K}, \ \boldsymbol{e}\boldsymbol{e}^T \bullet \boldsymbol{x}\boldsymbol{x}^T = \rho \right\}, \\ &\widetilde{G}_1 = \left\{ (\boldsymbol{x}, \boldsymbol{s})(\boldsymbol{x}, \boldsymbol{s}) \in \widetilde{G}_0 : \begin{array}{l} f_i(\boldsymbol{x}) = 0 \ (i = 1,2,3,4), \ g_{11}(\boldsymbol{x}) = 0, \ g_{12}(\boldsymbol{x}) = 0, \\ f_j(\boldsymbol{x}, \boldsymbol{s}) = 0 \ (j = 5,6), \ g_{13}(\boldsymbol{x}, \boldsymbol{s}) = 0, \ g_{14}(\boldsymbol{x}, \boldsymbol{s}) = 0 \end{array} \right\} \end{aligned}$$

to represent F as follows:

$$F = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \begin{array}{l} (\boldsymbol{x}, \boldsymbol{s}) \in \mathbb{K}, \ (\boldsymbol{x}, \boldsymbol{s})(\boldsymbol{x}, \boldsymbol{s})^T \in \widetilde{G}_1 \\ \text{for some } \ (x_5, x_6, x_7, x_8, s_1, s_2) \in \mathbb{R}^6 \end{array} \right\}.$$

Finally, we choose $\boldsymbol{H}_0 \in \mathbb{S}^{10}_+$ and $\boldsymbol{H}_1 \in \mathbb{S}^{10}_+ + \mathbb{N} \subset \mathbb{C}$ in a similar way to (5.8) so that $\widetilde{G}_0$ and $\widetilde{G}_1$ are represented as in (3.1) with $p = 1$. Conditions (A)', ($\widetilde{\text{B}}$) and ($\widetilde{\text{C}}$) are satisfied since $O \neq \boldsymbol{H}_0 \in \mathbb{S}^{10}_+$ and $\boldsymbol{H}_1 \in \mathbb{C}$, and Condition (D) since $\widetilde{G}_1$ is bounded.

We can apply the method mentioned above for decreasing the levels of hierarchy to QOP (5.1) in the previous section. First, we replace D by $D \bigcap \left(\mathbb{R}^r_+ \times \mathbb{R}^{m-r}\right)$ so that

$$0 \leq u_i \ (i = 1, 2, \ldots, r) \text{ if } \boldsymbol{u} \in D.$$

Next, introducing slack variable vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_r) \in \mathbb{R}^r$, we add constraints $u_i + v_i = 1, v_i \geq 0$ ($i = 1, 2, \ldots, r$) to QOP (5.1), and rewrite QOP (5.1) as

$$\begin{array}{ll} \text{minimize} & \boldsymbol{u}^T \boldsymbol{Q}_0 \boldsymbol{u} + 2\boldsymbol{c}^T \boldsymbol{u} \\ \text{subject to} & (\boldsymbol{u}, \boldsymbol{v}) \in D \times \mathbb{R}^r_+, \ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} = \boldsymbol{0}, \ u_i + v_i = 1 \ (i = 1, 2, \ldots, r), \\ & u_i v_i = 0 \ (i = 1, 2, \ldots, r). \end{array}$$

Now the binary condition $u_i(1 - u_i) = 0$ has been replaced by the complemetarity condition $u_i v_i = 0$ with the additional constraints $u_i + v_i = 1$ and $v_i \geq 0$ ($i = 1, 2, \ldots, r$). Redefining

$$\mathbb{K} = \left\{ (u_0, u_0 \boldsymbol{u}, u_0 \boldsymbol{v}) \in \mathbb{R}^{1+m+r} : u_0 \in \mathbb{R}_+, \ (\boldsymbol{u}, \boldsymbol{v}) \in D \times \mathbb{R}^r_+ \right\},$$

we replace QOP (5.2) by

$$\begin{array}{ll} \text{minimize} & \boldsymbol{u}^T \boldsymbol{Q}_0 \boldsymbol{u} + 2\boldsymbol{c}^T u_0 \boldsymbol{u} \\ \text{subject to} & (u_0, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{K}, \ u_0 = 1, \\ & \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} u_0 = \boldsymbol{0}, \ u_i + v_i - u_0 = 0 \ (i = 1, 2, \ldots, r), \\ & \displaystyle\sum_{i=1}^{r} u_i v_i = 0, \end{array}$$

and replace $\widetilde{G}_0$ and $\widetilde{G}_1$ by

$$\widetilde{G}_0 = \left\{(u_0, \boldsymbol{u}, \boldsymbol{v})(u_0, \boldsymbol{u}, \boldsymbol{v})^T : (u_0, \boldsymbol{u}, \boldsymbol{v}) \in \mathbb{K},\ u_0^2 = 1\right\},$$

$$\widetilde{G}_1 = \left\{ (u_0, \boldsymbol{u}, \boldsymbol{v})(u_0, \boldsymbol{u}, \boldsymbol{v})^T \in \widetilde{G}_0 : \begin{array}{l} \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}u_0 = \boldsymbol{0}, \\ u_i + v_i - u_0 = 0\ (i = 1, 2, \ldots, r), \\ u_i v_i = 0\ (i = 1, 2, \ldots, r) \end{array} \right\}.$$

Finally, we choose appropriate $\boldsymbol{H}_0 \in \mathbb{S}_+^{1+m+r}$ and $\boldsymbol{H}_1 \in \mathbb{S}_+^{1+m+r} + \mathbb{N} \subset \mathbb{C}$ to represent $\widetilde{G}_0$ and $\widetilde{G}_1$ as in (3.1) with $p = 1$.

5.3. QOPs involving a variable vector in a sphere. Let ρ be a positive number, $\boldsymbol{A}$ a $q \times m$ matrix and $\boldsymbol{I}$ the $m \times m$ identity matrix. We consider the set F of $\boldsymbol{u} \in \mathbb{R}^m$ satisfying

$$\boldsymbol{u} \in \mathbb{R}_+^m \ (\text{or } \boldsymbol{u} \geq \boldsymbol{0}),\ \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T = \rho,\ \ \boldsymbol{A}\boldsymbol{u} \leq \boldsymbol{0}. \tag{5.9}$$

Introduce a variable vector $\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}_+^{m+q}$, where $\boldsymbol{s} \in \mathbb{R}^q$ serves as a slack vector for the inequality $\boldsymbol{A}\boldsymbol{u} \leq \boldsymbol{0}$, and matrices $\boldsymbol{H}_\ell$ $(\ell = 0, 1)$ such that

$$\boldsymbol{H}_0 = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \in \mathbb{S}_+^{m+q},\ \boldsymbol{H}_1 = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{I} \end{pmatrix}^T \begin{pmatrix} \boldsymbol{A} & \boldsymbol{I} \end{pmatrix} \in \mathbb{S}_+^{m+q}.$$

Let $n = m + q$. Define $\widetilde{G}_0$ and $\widetilde{G}_1$ as in (3.1) with $p = 1$. Then, we can rewrite F as

$$F = \left\{ \boldsymbol{u} \in \mathbb{R}_+^m :\ \ \boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}_+^n,\ \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_1 \ \text{ for some } \boldsymbol{s} \in \mathbb{R}^q \right\}.$$

Apparently, Condition (A)' is satisfied, so is Condition $(\widehat{\mathrm{B}})$ because both $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are positive semidefinite. Since $\widetilde{G}_1$ is bounded, Condition (D) holds.

Now we consider the question whether the homogeneous inequality $\boldsymbol{A}\boldsymbol{u} \leq \boldsymbol{0}$ could be replaced by an inhomogeneous inequality $\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0}$ with a nonzero $\boldsymbol{b} \in \mathbb{R}^q$ in the discussions above. Notice that $\boldsymbol{A}\boldsymbol{u} \leq \boldsymbol{0}$ can be replaced by $\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}\boldsymbol{e}^T\boldsymbol{u} \leq \boldsymbol{0}$, and the coefficient $\boldsymbol{e}^T\boldsymbol{u}$ of $\boldsymbol{b}$ varies from $\sqrt{\rho}$ through $\sqrt{m\rho}$. But, the inequality cannot be replaced by an inhomogeneous inequality. For this, we need a different formulation, which can be described as

$$\begin{array}{l} \boldsymbol{x} = (u_0, \boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}_+^{1+m+q},\ u_0^2 = 1,\ (-\boldsymbol{b}u_0 + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{s})^T(-\boldsymbol{b}u_0 + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{s}) = 0, \\ \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T - \rho u_0^2 = 0\ \left(\text{or } \rho u_0^2 - \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T = 0\right). \end{array}$$

If we define

$$\begin{array}{rcl} \boldsymbol{H}_0 & = & \begin{pmatrix} 1 & \boldsymbol{0}^T & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} & \boldsymbol{O}^T \\ \boldsymbol{0} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix},\ \boldsymbol{H}_1 = \begin{pmatrix} \boldsymbol{b}^T\boldsymbol{b}^2 & -\boldsymbol{b}^T\boldsymbol{A} & -\boldsymbol{b}^T \\ -\boldsymbol{A}^T\boldsymbol{b} & \boldsymbol{A}^T\boldsymbol{A} & \boldsymbol{A}^T \\ -\boldsymbol{b} & \boldsymbol{A} & \boldsymbol{I} \end{pmatrix}, \\ \boldsymbol{H}_2 & = & \begin{pmatrix} -\rho^2 & \boldsymbol{0}^T & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{O}^T \\ \boldsymbol{0} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \left(\text{or } \boldsymbol{H}_2 = \begin{pmatrix} \rho^2 & \boldsymbol{0}^T & \boldsymbol{0}^T \\ \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{O}^T \\ \boldsymbol{0} & \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \right), \end{array} \tag{5.10}$$

and $\widetilde{G}_\ell$ $(\ell = 0, 1, 2)$ as in (3.1) with $p = 2$, then we have

$$\begin{array}{l} \left\{\boldsymbol{u} \in \mathbb{R}_+^m : \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T = \rho,\ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0}\right\} \\ = \left\{ \boldsymbol{u} \in \mathbb{R}_+^m :\ \ \boldsymbol{x} = (u_0, \boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}_+^{1+m+q},\ \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_2 \text{ for some } (u_0, \boldsymbol{s}) \in \mathbb{R}^{1+q} \right\}. \end{array}$$

However, the inequality

$$\begin{array}{c}\rho - \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T \geq 0 \text{ if } \ \boldsymbol{u} \in \mathbb{R}^m_+ \ \text{ and } \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0} \\ (\text{or } \ \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T - \rho \geq 0 \text{ if } \ \boldsymbol{u} \in \mathbb{R}^m_+ \ \text{ and } \ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0} \ )\end{array}$$

is required for Condition $(\widetilde{\mathrm{B}})$ to be satisfied. In other words, the polyhedral set $\{\boldsymbol{u} \in \mathbb{R}^m_+ : \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0}\}$ needs to be inside (or outside) of the ball $\left\{\boldsymbol{u} \in \mathbb{R}^m : \|\boldsymbol{u}\| \leq \sqrt{\rho}\right\}$, touching the ball only at its boundary points. This requirement may be regarded as too strong. It seems difficult to formulate the inhomogeneous problems in terms of our framework with $\mathbb{K} = \mathbb{R}^n_+$.

Now, we formulate the inhomogeneous case as follows

$$\begin{array}{l}\mathbb{K} = \text{cl cone } \left\{\boldsymbol{x} = (u_0, u_0\boldsymbol{u}, u_0\boldsymbol{s}) \in \mathbb{R}^{1+m+q} : (u_0, \boldsymbol{u}, \boldsymbol{s}) \in \mathbb{R}^{1+m+q}_+, \ \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T = \rho\right\}, \\ \boldsymbol{x} = (u_0, \boldsymbol{u}, \boldsymbol{s}) \in \mathbb{K}, \ u_0^2 = 1, \ (-\boldsymbol{b}u_0 + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{s})^T(-\boldsymbol{b}u_0 + \boldsymbol{A}\boldsymbol{u} + \boldsymbol{s}) = 0, \\ \boldsymbol{H}_0 \ \text{ and } \boldsymbol{H}_1 \ \text{ as in (5.10), } \widetilde{G}_0 \ \text{ and } \widetilde{G}_1 \ \text{ as in (3.1) with } p = 1.\end{array}$$

Then, we have

$$\begin{array}{l}\left\{\boldsymbol{u} \in \mathbb{R}^m_+ : \boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T = \rho, \ \boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0}\right\} \\ = \left\{\boldsymbol{u} \in \mathbb{R}^m_+ : \ \boldsymbol{x} = (u_0, \boldsymbol{u}, \boldsymbol{s}) \in \widetilde{G}_1 \text{ for some } (u_0, \boldsymbol{s}) \in \mathbb{R}^{1+q} \ \right\}.\end{array}$$

Condition (A)' is obviously satisfied, so is Condition $(\widehat{\mathrm{B}})$ by $\boldsymbol{H}_1 \in \mathbb{S}^{1+m+q}_+$. We also see that

$$\begin{aligned}\widetilde{L}_1 &= \left\{\boldsymbol{d}\boldsymbol{d}^T : \boldsymbol{d} = (v_0, \boldsymbol{v}, \boldsymbol{t}) \in \mathbb{K}, \ \boldsymbol{H}_0 \bullet \boldsymbol{d}\boldsymbol{d}^T = 0, \ \boldsymbol{H}_1 \bullet \boldsymbol{d}\boldsymbol{d}^T = 0\right\} \\ &= \left\{\boldsymbol{d}\boldsymbol{d}^T : \boldsymbol{d} = (0, \boldsymbol{v}, \boldsymbol{t}) \in \mathbb{K}, \ \boldsymbol{A}\boldsymbol{u} + \boldsymbol{s} = \boldsymbol{0}\right\} \\ &= \{\boldsymbol{O}\}.\end{aligned}$$

Here the last identity follows from the definition of $\mathbb{K}$ above. Thus, (D) holds.

5.4. A QOP involving a copositive matrix. The 5×5 matrix

$$\boldsymbol{M} = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

is known to be a copositive matrix, which is not a sum of any pair of a positive semidefinite matrix and a nonnegative matrix [12]. See also [4]. The associated quadratic form is represented as

$$\begin{aligned}\boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T &= (x_1 - x_2 + x_3 - x_4 + x_5)^2 + 4x_2x_5 + 4x_1(x_4 - x_5) \\ &= (x_1 - x_2 + x_3 + x_4 - x_5)^2 + 4x_2x_4 + 4x_3(x_5 - x_4),\end{aligned}$$

which shows that $\boldsymbol{M}$ is copositive. In fact, if $\boldsymbol{x} \geq \boldsymbol{0}$ and $x_4 \geq x_5$ then $\boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T$ is nonnegative by the first representation, and if $\boldsymbol{x} \geq \boldsymbol{0}$ and $x_5 \geq x_4$, then it is nonnegative by the second.

We consider the set F of $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_+$ satisfying

$$x_1 = 1,\ \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0,\ 2x_4 + x_5 \leq 3 \tag{5.11}$$

or equivalently,

$$x_1 = 1, \tag{5.12}$$
$$(x_1 - x_2 + x_3 - x_4 + x_5)^2 + 4x_2x_5 + 4x_1(x_4 - x_5) = 0, \tag{5.13}$$
$$3x_1 - 2x_4 - x_5 - x_6 = 0 \tag{5.14}$$

for a slack variable $x_6 \geq 0$. We rewrite these constraints as

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_6) \in \mathbb{R}^6_+,\ \boldsymbol{H}_0 \bullet \boldsymbol{x}\boldsymbol{x}^T = 1,\ \boldsymbol{H}_1 \bullet \boldsymbol{x}\boldsymbol{x}^T = 0,\ \boldsymbol{H}_2 \bullet \boldsymbol{x}\boldsymbol{x}^T = 0,$$

where

$$\boldsymbol{e}_1 = (1, 0, 0, 0, 0, 0) \in \mathbb{R}^6,\ \boldsymbol{a} = (3, 0, 0, -2, -1, -1) \in \mathbb{R}^6_+$$
$$\boldsymbol{H}_0 = \boldsymbol{e}_1\boldsymbol{e}_1^T \in \mathbb{S}^6_+,\ \boldsymbol{H}_1 = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0}^T & 0 \end{pmatrix} \in \mathbb{C},\ \boldsymbol{H}_2 = \boldsymbol{a}\boldsymbol{a}^T \in \mathbb{S}^6_+.$$

Define $\widetilde{G}_\ell$ $(\ell = 0, 1, 2)$ as in (3.1) with $n = 6$, $\rho = 1$ and $p = 2$. Then, we obtain

$$F = \left\{ (x_1, x_2, \ldots, x_5) \in \mathbb{R}^5 : \boldsymbol{x} = (x_1, x_2, \ldots, x_6) \in \mathbb{R}^6_+,\ \boldsymbol{x}\boldsymbol{x}^T \in \widetilde{G}_2 \text{ for some } x_6 \in \mathbb{R}_+ \right\}.$$

Since $\boldsymbol{O} \neq \boldsymbol{H}_0 \in \mathbb{S}^6_+$, $\boldsymbol{H}_1 \in \mathbb{C}$ and $\boldsymbol{H}_2 \in \mathbb{S}^6_+$, Conditions (A)' and ($\widehat{\text{B}}$) with $p = 2$ are satisfied. To confirm that Condition (D) with $p = 2$ holds, we let $\boldsymbol{O} \neq \boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_2$. Then, $\boldsymbol{x} = \boldsymbol{d} \in \mathbb{R}^6$ satisfies $x_1 = 0$, (5.13) and (5.14). Thus, $\boldsymbol{d}$ is of the form $\boldsymbol{d} = \delta(0, 1, 1, 0, 0, 0) \in \mathbb{R}^6_+$ for some $\delta > 0$. Let $\boldsymbol{x}(\nu) = (1, 1 + \sqrt{\nu}\delta, \sqrt{\nu}\delta, 0, 0, 3) \in \mathbb{R}^6_+$ for every $\nu \geq 0$. Then, $\boldsymbol{x}(\nu) \in \mathbb{R}^6_+$ satisfies (5.12), (5.13) and (5.14). Thus, $\boldsymbol{x}(\nu)\boldsymbol{x}(\nu)^T \in \widetilde{G}_2$. We observe that $(1/\sqrt{\nu}, \boldsymbol{x}(\nu)/\sqrt{\nu}) \to (0, \boldsymbol{d})$ as $\nu \to \infty$. Therefore $\boldsymbol{d}\boldsymbol{d}^T$ is an asymptotic unbounded direction of $\widetilde{G}_2$.

Now, we consider the case where the set $F \subset \mathbb{R}^5_+$ is given by

$$x_1 = 1,\ \boldsymbol{M} \bullet \boldsymbol{x}\boldsymbol{x}^T = 0,\quad x_5 \leq 3$$

instead of (5.11). Note that the last inequality $2x_4 + x_5 \leq 3$ in (5.11) is replaced by $x_5 \leq 3$. If we replace (5.14) by

$$3x_1 - x_5 - x_6 = 0 \tag{5.15}$$

and $\boldsymbol{a} = (3, 0, 0, -2, -1, -1) \in \mathbb{R}^6_+$ by $\boldsymbol{a} = (3, 0, 0, 0, 1, -1) \in \mathbb{R}^6_+$, all the previous discussions remain valid, except the one on Condition (D). In this case, $\boldsymbol{d} \in \mathbb{R}^6_+$ such that $\boldsymbol{d}\boldsymbol{d}^T \in \widetilde{L}_2$ is characterized by $\boldsymbol{x} = \boldsymbol{d}$ satisfying $x_1 = 0$, (5.13) and (5.15). For example, $\boldsymbol{d}\boldsymbol{d}^T$ with $\boldsymbol{d} = (0, 0, 1, 1, 0, 0)$ lies in $\widetilde{L}_2$. But $\boldsymbol{d}\boldsymbol{d}^T$ cannot be an asymptotic unbounded direction of $\widetilde{G}_2$. To verify this, assume on the contrary that there is a sequence $\left\{ (\mu^s, \boldsymbol{u}^s(\boldsymbol{u}^s)^T) \in \mathbb{R}_+ \times \widetilde{G}_2 : s = 1, 2, \ldots \right\}$ such that $\|\boldsymbol{u}^s\| \to \infty$ and $(\sqrt{\mu^s}, \sqrt{\mu^s}\boldsymbol{u}^s) \to (0, \boldsymbol{d})$ as $s \to \infty$. From $\boldsymbol{u}^s(\boldsymbol{u}^s)^T \in \widetilde{G}_2$, we have $u^s_1 = 1$ and $u^s_4 = u^s_5$. This implies that $d_4 = d_5$, which is a contradiction to $d_4 = 1$ and $d_5 = 0$. Thus, we have shown that $\boldsymbol{d}\boldsymbol{d}^T$ is not an asymptotic unbounded direction of $\widetilde{G}_2$.

6. Concluding Remarks. The reformulation of a class of QOPs into CPPs has been proposed under two sets of sufficient conditions. The key idea has been constructing a hierarchical structure into their feasible regions (see (3.1)) and imposing the copositivity condition (Condition (B) and (B)') recursively on each level of the constraints of hierarchical structure. Although the class of QOPs that can be reformulated into CPPs using this idea may seem limited, they do include various QOPs as seen in Section 5. When it is applied to a QOP with linear constraints in continuous nonnegative variables and binary variables, the resulting equivalent CPP involves just three equality constraints (or even only two equality constraints, see the last paragraph of Section 5.2). This property is a distinctive feature of our QOP model for exact CPP relaxation, and is expected to be effectively utilized to develop new and powerful numerical methods for such a QOP.

On the other hand, the reformulation of a QOP in Section 5.4, which simultaneously involves a sphere constraint $\boldsymbol{I} \bullet \boldsymbol{u}\boldsymbol{u}^T = \rho$ and an inhomogeneous inequality constraint $\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b} \leq \boldsymbol{0}$ in $\boldsymbol{u} \in \mathbb{R}^m_+$, into an equivalent CPP relaxation has not been successful. The main reason for this is that each QOP in our class is allowed to have one inhomogeneous equality for the construction of the hierarchy of copositivity. When the proposed idea is considered to be applied to a wider class of applications in practice, this issue needs to be resolved. This will be a subject of future study for generalizing the QOP model.

We were informed of the paper by Peña, Vera and Zuluaga [22] after we submitted this paper to SIAM Journal on Optimization. They applied a canonical convexification procedure to a general polynomial optimization problem (abbreviated by POP) with nonnegative variables, and presented a linear optimization problem over the cone of completely positive d-forms equivalent to the POP. Among the assumptions they imposed on the POP, they introduced two conditions, which are essentially equivalent to (but a little bit stronger than) our hierarchy of copositivity condition (Condition $(\widetilde{B}) + (\widetilde{C})$) and our condition on the asymptotic unbounded directions of $\widetilde{G}$ (Condition (D)), respectively.

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