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Naohiko Arima, Sunyoung Kim
and Masakazu Kojima

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B-471 Extension of Completely Positive Cone Relaxation to Polynomial Optimization

Naohiko Arima[⋆], Sunyoung Kim[†] and Masakazu Kojima[‡]

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Abstract. We propose the moment cone relaxation for a class of polynomial optimization problems (POPs) to extend the results on the completely positive cone programming relaxation for the quadratic optimization (QOP) model by Arima, Kim and Kojima. The moment cone relaxation is constructed to take advantage of sparsity of the POPs, so that efficient numerical methods can be developed in the future. We establish the equivalence between the optimal value of the POP and that of the moment cone relaxation under conditions similar to the ones assumed in the QOP model. The proposed method is compared with the canonical convexification procedure recently proposed by Peña, Vera and Zuluaga for POPs. The moment cone relaxation is theoretically powerful, but numerically intractable. For tractable numerical methods, the doubly nonnegative cone relaxation is derived from the moment cone relaxation. Exploiting sparsity in the doubly nonnegative cone relaxation and its incorporation into Lasserre's semidefinite relaxation are briefly discussed.

Key words. Moment cone relaxation, doubly nonnegative cone relaxation, polynomial optimization, copositive programming, completely positive programming.

AMS Classification. 90C20, 90C25, 90C26

⋆ Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152-8552 Japan.
`arima.n.ab@m.titech.ac.jp`

† Department of Mathematics, Ewha W. University, 11-1 Dahyun-dong, Sudaemoon-gu, Seoul 120-750 Korea. The research was supported by NRF 2012-R1A1A2-038982 and NRF 2010-000-8784.
`skim@ewha.ac.kr`

‡ Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152-8552 Japan, and Research and Development Initiative & JST CREST, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, Tokyo 112-8551 Japan. This research was supported by Grant-in-Aid for Scientific Research (B) 22310089 and the Japan Science and Technology Agency (JST), the Core Research of Evolutionary Science and Technology (CREST) research project.
`kojima@is.titech.ac.jp`

1 Introduction

The copositive programming (CP) and completely positive programming (CPP) relaxation [1, 2, 3, 4, 5, 9, 10, 13, 17, 18] for quadratic optimization problems (QOPs) have attracted considerable attention in recent years. The class of QOPs considered by Burer in [4] was binary and continuous non-convex QOPs with linear constraints, and a QOP with an additional constraint $\boldsymbol{u} \in D$ in its variable vector $\boldsymbol{u}$, where D is a closed (not necessarily convex) set, was represented in a CPP by Eichfelder and Povh [10] (see also [8]), extending Burer's results. Burer and Dong [6] generalized the standard CPP relaxation over nonnegative orthant to ones over second order cone and positive semidefinite cone to cover a general class of quadratically constrained quadratic programs. More recently, it was shown in [1] that a QOP model with quadratic constraints could be reformulated as a CPP under the hierarchy of copositivity and zeros at infinity conditions, which had also been used in [16]. All of these results show that the proposed CP and CPP relaxations are exact for the given QOP, that is, the optimal value of the CPP relaxation is equivalent to that of the given QOP.

For polynomial optimization problems (POPs), which include QOPs, semidefinite programming (SDP) relaxations proposed by [14, 15] have been very popular as solution methods. Noting the CPP relaxations are stronger than SDP relaxations for QOPs, it is natural to ask whether the results on the CP and CPP relaxations for QOPs can be extended to a class of POPs. Peña, Vera and Zuluaga [16] extended the CPP relaxation to POPs and proposed a canonical convexification procedure for POPs under the hierarchy of copositivity and zeros at infinity conditions. Their procedure was focused on the theoretical reformulation of the POP into a generalization of the CPP.

The main goal of this paper is to propose the *moment cone relaxation* for a class of POPs as an extension of the CPP relaxation given in [1], and present a method to exploit sparsity of POPs in the doubly nonnegative cone relaxation, a further relaxation of the moment cone relaxation. Let $\mathbb{R}[\boldsymbol{x}]$ be the set of real-valued multivariate polynomials in n variables $x_1, \ldots, x_n \in \mathbb{R}$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. As a theoretical framework for the moment cone relaxation, we consider the following POP:

$$\text{minimize} \quad \psi(\boldsymbol{x}) \quad \text{subject to} \quad h_0(\boldsymbol{x}) = 1, \; h_j(\boldsymbol{x}) = 0 \; (j \in J), \; \boldsymbol{x} \in \mathbb{L}, \tag{1}$$

where $J = \{1, \ldots, \ell\}$, $J_0 = \{0\} \bigcup J = \{0, 1, \ldots, \ell\}$, $\psi, \; h_j \in \mathbb{R}[\boldsymbol{x}]$ $(j \in J_0)$ and $\mathbb{L}$ is a closed (not necessarily convex) cone in $\mathbb{R}^n$. This model is an extension of the standard QOP model [3] of minimizing a quadratic form over the simplex represented as $\left\{ \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}, \; \left(\sum_{i=1}^n x_i\right)^2 = 1 \right\}$ and the QOP model studied in [1, 2]. We assume throughout the paper that

$$\psi, \; h_j \in \mathbb{R}[\boldsymbol{x}] \; (j \in J_0) \text{ are homogeneous polynomials with some degree } \tau \geq 1. \tag{2}$$

Here $f \in \mathbb{R}[\boldsymbol{x}]$ is called a homogeneous polynomial with degree τ if

$$f(\lambda \boldsymbol{x}) = \lambda^\tau f(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}.$$

We present the *moment cone relaxation* for POP (1) as an extension of the CPP relaxation, and show under certain conditions that the optimal value of the POP (1) coincides with that of the moment cone relaxation of (1). The conditions include the hierarchy of copositivity and a variation of the zeros at infinity conditions introduced in [16] and later used in [1]. We note that a simple copositivity condition

$$h_j(\boldsymbol{x}) \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{L}, \tag{3}$$

is a stronger version of the hierarchy of copositivity, and

$$\boldsymbol{x} = \boldsymbol{0} \text{ if } \boldsymbol{x} \in \mathbb{L} \text{ and } h_j(\boldsymbol{x}) = 0 \ (j \in J_0) \tag{4}$$

is a stronger version of the zero at infinity condition. Condition (3) is not so strong theoretically because $\psi(\boldsymbol{x})$ can always be replaced by $h_0(\boldsymbol{x})\psi(\boldsymbol{x})$ and $h_j(\boldsymbol{x})$ by $h_j(\boldsymbol{x})^2$ $(j \in J_0)$ in POP (1) to satisfy both (2) and (3). This, however, may destroy sparsity of the polynomials in the problem. Condition (4), together with (2), requires that the feasible region of POP (1) is bounded, while the zeros at infinity, a weaker condition, allows that the feasible region is unbounded.

We compare the proposed moment cone relaxation with the canonical convexification procedure proposed by Peña, Vera and Zuluaga [16] for POPs. They deal with POP of the form

$$\text{minimize} \quad \varphi(\boldsymbol{w}) \quad \text{subject to} \quad g_j(\boldsymbol{w}) = 0 \ (j \in J), \ \boldsymbol{w} = (w_1, \ldots, w_m) \in \mathbb{K}, \tag{5}$$

where $\mathbb{K}$ denotes a closed cone, $J = \{1, \ldots, \ell\}$ and $\varphi, \ g_j \in \mathbb{R}[\boldsymbol{w}]$ $(j \in J)$. We note that the homogenuity of the polynomials $\varphi, \ g_j \in \mathbb{R}[\boldsymbol{w}]$ $(j \in J)$ is not assumed in POP (5), but (5) is easily transformed into POP (1) satisfying the homogenuity condition (2) on the polynomials $\varphi, \ g_j \in \mathbb{R}[\boldsymbol{w}]$ $(j \in J)$ by introducing an auxiliary variable $w_0 \in \mathbb{R}_+$ fixed to 1, which corresponds to the equality constraint $h_0(w_0, \boldsymbol{w}) = 1$, with $\boldsymbol{x} = (w_0, \boldsymbol{w})$ in (1), and setting the cone $\mathbb{L} = \mathbb{R}_+ \times \mathbb{K}$.

For (homogeneous) QOPs, two different descriptions of the completely positive cone are known:

$$\text{the convex cone generated by } \left\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}^n_+\right\}$$

and

$$\left\{\sum_{p=1}^{q} \boldsymbol{x}_p\boldsymbol{x}_p^T : \boldsymbol{x}_p \in \mathbb{R}^n_+ \ (p = 1, \ldots, q), \ q \geq 0\right\}.$$

These two descriptions are equivalent. In [16] by Peña, Vera and Zuluaga, the completely positive cone in the former description is generalized to the cone of completely positive d-forms. The completely positive cone described as the latter can be generalized similarly. However, when their canonical convexification procedure is applied to nonhomogeneous POPs of the form (5), the two generalized descriptions *are different*. In particular, the latter is neither convex nor conic. On the one hand, the two descriptions remain equivalent in our homogeneous POP model (1) satisfying (2) (see Lemma 3.1). This is a fundamental and essential feature of our POP model, which makes it possible to

- allow a straightforward extension from the CPP relaxation for the QOP model [1] to POP (1),
- make the derivation of an equivalent moment cone relaxation of (1) simple,
- directly handle cases where the closed cone $\mathbb{L}$ is neither convex nor pointed,
- naturally take account of sparsity of the polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}]$ $(j \in J_0)$ in POP (1).

We note that the last advantage is important for developing efficient approximation of the moment cone relaxation problem in practice, such as the doubly nonnegative relaxation discussed in Section 6. More detailed comparison is included in Section 5.

POP (1) is quite general in that it includes various types of QOPs and POPs. We refer to the papers [1, 3, 4, 6, 8, 16, 17] for the QOPs and POPs that are easily transformed into POPs of the form (1) satisfying the required conditions for the equivalence to its moment cone relaxation. A popular choice for the closed cone $\mathbb{L}$ in (1) may be the Cartesian product of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}_+$ for some n_1 and n_2 satisfying $n = n_1 + n_2$. More generally, we can choose a second order cone, the vectorization of a positive semidefinite symmetric matrix cone and the vectorization of a cone of nonnegative symmetric matrices for $\mathbb{L}$. We also note that if $\mathbb{L}_1$ and $\mathbb{L}_2$ in $\mathbb{R}^n$ are cones, so are their intersection, union, difference, symmetric difference, and Minkowski sum.

The proposed moment cone relaxation for POP (1) is very powerful in theory, but it is quite difficult to implement the relaxation numerically. Even for the CPP relaxation of QOPs, neither effective nor efficient numerical methods have been developed. As a tractable numerical method, we extend the doubly nonnegative cone relaxation [11, 23] for the CPP problem to the moment cone relaxation of POP (1) with $\mathbb{L} = \mathbb{R}^n_+$. For the extension, we choose a collection of monomials (satisfying a certain symmetric property) that covers the monomials involved in $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}]$ $(j \in J)$. Then, the moment cone with $\mathbb{L} = \mathbb{R}^n_+$ and the collection induce a cone that lies in the doubly nonnegative cone (= the intersection of the positive semidefinite matrix cone and the cone of nonnegative matrices) in a symmetric matrix space. We also discuss how to exploit sparsity in the doubly nonnegative relaxation, and briefly shows that the idea of doubly nonnegative relaxation can easily be incorporated in Lasserre's SDP relaxation [14].

In Section 2, we summarize the notation and symbols. The illustrative example described in Section 2 is used throughout for better understanding of the discussions in this paper. The main results showing the equivalence of the optimal value of (1) and its moment cone relaxation are stated in Section 3, and their proofs in Section 4. In Section 5, we describe how to transform POP (5) into POP (1), and discuss some similarities and differences between the proposed moment cone relaxation and the canonical convexification procedure [16]. In Section 6, the doubly nonnegative cone relaxations for POPs are discussed. Concluding remarks are included in Section 7.

2 Preliminaries

2.1 Notation and symbols

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{Z}_+$ the set of nonnegative integers. We denote the i-th coordinate unit vector with the i-th element 1 and all other elements 0 as $\boldsymbol{e}_i \in \mathbb{R}^n$, and the vector of all elements 1 as $\mathbf{1} \in \mathbb{R}^n$. Let $|\boldsymbol{\beta}|_1 = \sum_{i=1}^n \beta_i$ for each $\boldsymbol{\beta} \in \mathbb{Z}^n_+$. $\mathbb{R}[\boldsymbol{x}]$ is the set of real-valued multivariate polynomials in n variables $x_1, \ldots, x_n \in \mathbb{R}$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Each polynomial $f \in \mathbb{R}[\boldsymbol{x}]$ is represented as $f(\boldsymbol{x}) = \sum_{\boldsymbol{\beta}\in\mathcal{H}} f_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}$, where $\mathcal{H} \subset \mathbb{Z}^n_+$ is a nonempty finite set, $f_{\boldsymbol{\beta}}$ ($\boldsymbol{\beta} \in \mathcal{H}$) are real coefficients, $\boldsymbol{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n_+$. We note that if $\mathbf{0} \in \mathcal{H}$ then $\boldsymbol{x}^{\mathbf{0}} = 1$ for any $\boldsymbol{x} \in \mathbb{R}^n$ and $f_{\mathbf{0}} \boldsymbol{x}^{\mathbf{0}}$ represents the constant term $f_{\mathbf{0}}$ of the polynomial $f \in \mathbb{R}[\boldsymbol{x}]$. The support of f is defined by $\mathrm{supp}(f) = \{\boldsymbol{\beta} \in \mathcal{H} : f_{\boldsymbol{\beta}} \neq 0\} \subset \mathbb{Z}^n_+$, and the degree of $f \in \mathbb{R}[\boldsymbol{x}]$ is defined by $\deg(f) = \max\{|\boldsymbol{\beta}|_1 : \boldsymbol{\beta} \in \mathrm{supp}(f)\}$. And, $\deg(\mathcal{H}) = \max\{|\boldsymbol{\beta}|_1 : \boldsymbol{\beta} \in \mathcal{H}\}$ for every nonempty finite subset $\mathcal{H}$ of $\mathbb{Z}^n_+$.

Let $\mathcal{H}$ be a nonempty finite subset of $\mathbb{Z}^n_+$. $|\mathcal{H}|$ stands for the number of elements of $\mathcal{H}$. $\mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ denotes the set of real-valued multivariate polynomials in $x_1, \ldots, x_n \in \mathbb{R}$ whose supports belong to $\mathcal{H}$; *i.e.*, $\mathbb{R}[\boldsymbol{x}, \mathcal{H}] = \{f \in \mathbb{R}[\boldsymbol{x}] : \mathrm{supp}(f) \subset \mathcal{H}\}$. Let $\mathbb{R}^{\mathcal{H}}$ denote the $|\mathcal{H}|$-dimensional Euclidean space whose coordinate are indexed by $\boldsymbol{\beta} \in \mathcal{H}$. For $A \subset \mathbb{R}^{\mathcal{H}}$, conv A denotes the convex hull of A, cone A the cone generated by A, and closure A the closure of A; hence closure conv A is the closure of the convex hull of A. Each vector of $\mathbb{R}^{\mathcal{H}}$ with elements $z_{\boldsymbol{\beta}}$ ($\boldsymbol{\beta} \in \mathcal{H}$) is denoted as $(z_{\boldsymbol{\beta}} : \mathcal{H})$. We assume that $(z_{\boldsymbol{\beta}} : \mathcal{H})$ is a column column vector when it is multiplied by a matrix. If $\boldsymbol{x} \in \mathbb{R}^n$, $(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H})$ denotes the $|\mathcal{H}|$-dimensional (column) vector with elements $z_{\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\beta}}$ ($\boldsymbol{\beta} \in \mathcal{H}$). Using the symbols introduced here, we frequently write a polynomial $f \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ as $f(\boldsymbol{x}) = (f_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H})$ for some $(f_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}}$, where $(f_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H})$ denotes the inner product $\sum_{\boldsymbol{\beta}\in\mathcal{H}} f_{\beta} \boldsymbol{x}^{\boldsymbol{\beta}}$ of $(f_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}}$ and $(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}}$.

The following notation and symbols are used in Section 6 where the doubly nonnegative cone relaxation is discussed. Let $\mathcal{F}$ be a nonempty finite subset of $\mathbb{Z}^n_+$. $\mathbb{S}^{\mathcal{F}}$ denotes the linear space of $|\mathcal{F}| \times |\mathcal{F}|$ symmetric matrices with elements $w_{\boldsymbol{\alpha\beta}}$ ($\boldsymbol{\alpha} \in \mathcal{F},\ \boldsymbol{\beta} \in \mathcal{F}$). Each matrix of $\mathbb{S}^{\mathcal{F}}$ is written as $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F})$. If $\boldsymbol{x} \in \mathbb{R}^n$, $(\boldsymbol{x}^{\boldsymbol{\alpha}} : \mathcal{F})(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{F})^T$ is a rank-1 symmetric matrix with elements $w_{\boldsymbol{\alpha\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ ($\boldsymbol{\alpha} \in \mathcal{F},\ \boldsymbol{\beta} \in \mathcal{F}$) in $\mathbb{S}^{\mathcal{F}}$, which is denoted by $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \square\mathcal{F})$. Here $(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{F})^T$ denotes the row vector obtained by taking the transpose of the column vector $(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{F})$.

2.2 An illustrative example

We consider a polynomial optimization problem

$$\begin{array}{ll} \text{minimize} & x_1^4 + 2x_1^2 x_2^2 - 4x_3^4 \\ \text{subject to} & x_1^4 + x_2^4 + x_3^4 = 1,\ x_1 x_2 - x_3^2 \geq 0,\ x_i \geq 0\ (i = 1, 2, 3). \end{array} \tag{6}$$

By introducing a slack variable $x_4 \in \mathbb{R}$ and a variable vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, we convert the problem into

$$\text{minimize} \quad \psi(\boldsymbol{x}) \quad \text{subject to} \quad h_0(\boldsymbol{x}) = 1,\ h_1(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \mathbb{R}^4_+, \tag{7}$$

where $\psi,\ h_0,\ h_1 \in \mathbb{R}[\boldsymbol{x}]$ are defined by

$$\psi(\boldsymbol{x}) = x_1^4 + 2x_1^2x_2^2 - 4x_3^4,\ h_0(\boldsymbol{x}) = x_1^4 + x_2^4 + x_3^4,\ h_1(\boldsymbol{x}) = (x_1x_2 - x_3^2 - x_4^2)^2.$$

In addition to condition (2) with $\tau = 4$, the problem (7) satisfies conditions (3) and (4) with $J_0 = \{0,1\}$ and $\mathbb{L} = \mathbb{R}^4_+$. This problem serves as an illustrative example in the subsequent discussions.

We see that

$$\begin{aligned}
&\deg(\varphi) = 4,\ \deg(h_0) = 4,\ \deg(h_1) = 4,\\
&\mathrm{supp}(\psi) = \left\{ \begin{pmatrix}4\\0\\0\\0\end{pmatrix} \begin{pmatrix}2\\2\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\4\\0\end{pmatrix} \right\},\ \mathrm{supp}(h_0) = \left\{ \begin{pmatrix}4\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\4\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\4\\0\end{pmatrix} \right\},\\
&\mathrm{supp}(h_1) = \left\{ \begin{pmatrix}2\\2\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\2\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\2\end{pmatrix} \begin{pmatrix}0\\0\\4\\0\end{pmatrix} \begin{pmatrix}0\\0\\2\\2\end{pmatrix} \begin{pmatrix}0\\0\\0\\4\end{pmatrix} \right\}.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{H}_{\min} &= \mathrm{supp}(\psi) \bigcup \mathrm{supp}(h_0) \bigcup \mathrm{supp}(h_1)\\
&= \left\{ \begin{pmatrix}4\\0\\0\\0\end{pmatrix} \begin{pmatrix}2\\2\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\2\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\2\end{pmatrix} \begin{pmatrix}0\\4\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\4\\0\end{pmatrix} \begin{pmatrix}0\\0\\2\\2\end{pmatrix} \begin{pmatrix}0\\0\\0\\4\end{pmatrix} \right\}.
\end{aligned}$$

Then, we can regard $\psi,\ h_0,\ h_1 \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ for any $\mathcal{H} \supset \mathcal{H}_{\min}$. For example, if we take $\mathcal{H} = \mathcal{H}_{\min}$, $\psi \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ is represented as follows:

$$\begin{array}{lcl}
(\psi_{\boldsymbol{\beta}} : \mathcal{H}) &=& \left(\psi_{(4000)}, \psi_{(2200)}, \psi_{(1120)}, \psi_{(1102)}, \psi_{(0400)}, \psi_{(0040)}, \psi_{(0022)}, \psi_{(0004)}\right),\\
&=& (1, 2, 0, 0, 0, -4, 0, 0) \in \mathbb{R}^{\mathcal{H}},\\
(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) &=& (x_1^4, x_1^2x_2^2, x_1x_2x_3^2, x_1x_2x_4^2, x_2^4, x_3^4, x_3^2x_4^2, x_4^4) \in \mathbb{R}^{\mathcal{H}},\\
\psi(\boldsymbol{x}) &=& (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}].
\end{array} \tag{8}$$

3 Main results

We consider POP (1) satisfying condition (2). Recall that $\mathbb{L}$ denotes a closed (but not necessarily convex) cone. Let T_* denote the feasible region of POP (1);

$$T_* = \{\boldsymbol{x} \in \mathbb{L} : h_0(\boldsymbol{x}) = 1,\ \boldsymbol{h}_j(\boldsymbol{x}) = 0\ (j \in J)\}.$$

Condition (2) can be restated as

$$\begin{aligned}
&\psi(\lambda\boldsymbol{x}) = \lambda^\tau \psi(\boldsymbol{x}),\ h_j(\lambda\boldsymbol{x}) = \lambda^\tau h_j(\boldsymbol{x})\ (j \in J_0)\\
&\quad \text{for some integer } \tau \geq 1, \text{ every } \boldsymbol{x} \in \mathbb{R}^n \text{ and every } \lambda \in \mathbb{R}_+.
\end{aligned} \tag{9}$$

Let $\mathcal{H}_{\min} = \text{supp}(\psi) \bigcup \left(\bigcup_{j \in J_0} \text{supp}(h_j) \right)$. Then, condition (9) is equivalent to

$$|\boldsymbol{\beta}|_1 \quad = \quad \tau \text{ for some positive integer } \tau \geq 1 \text{ and every } \boldsymbol{\beta} \in \mathcal{H}_{\min}. \tag{10}$$

Let $\mathcal{H}_{\max} = \left\{ \boldsymbol{\beta} \in \mathbb{Z}^n_+ : |\boldsymbol{\beta}|_1 = \tau \right\}$. Choose $\mathcal{H} \subset \mathbb{Z}^n_+$ such that $\mathcal{H}_{\min} \subset \mathcal{H} \subset \mathcal{H}_{\max}$. Then, the polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}]$ $(j \in J_0)$ is written as

$$\psi(\boldsymbol{x}) = (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}),\ h_j(\boldsymbol{x}) = ((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) \ (j \in J_0),$$

for some $(\psi_{\boldsymbol{\beta}} : \mathcal{H}),\ ((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}}$ $(j \in J_0)$. Let

$$\begin{aligned} \widetilde{T}(\mathcal{H}) &= \left\{ (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}} : \boldsymbol{x} \in T_* \right\} \\ &= \left\{ (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}} : \begin{array}{c} \boldsymbol{x} \in \mathbb{L},\ ((h_0)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) = 1, \\ ((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) = 0 \ (j \in J) \end{array} \right\}. \end{aligned}$$

Then, we can rewrite POP (1) as

$$\text{minimize} \quad (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) \quad \text{subject to} \quad (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widetilde{T}(\mathcal{H}).$$

Since the objective function is linear with respect to $(z_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}}$, the problem above is equivalent to

$$\text{minimize} \quad (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) \quad \text{subject to} \quad (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \text{conv}\, \widetilde{T}(\mathcal{H}). \tag{11}$$

In the case of POP (7), we see that

$$\begin{aligned} \widetilde{T}(\mathcal{H}) &= \left\{ \begin{array}{l} (x_1^4, x_1^2x_2^2, x_1x_2x_3^2, x_1x_2x_4^2, x_2^4, x_3^4, x_3^2x_4^2, x_4^4) : \\ \qquad\qquad h_0(\boldsymbol{x}) = x_1^4 + x_2^4 + x_3^4 = 1, \\ \qquad\qquad h_1(\boldsymbol{x}) = (x_1x_2 - x_3^2 - x_4^2)^2 = 0,\ \boldsymbol{x} \in \mathbb{R}^4_+ \end{array} \right\}, \\ (z_{\boldsymbol{\beta}} : \mathcal{H}) &= \left(z_{(4000)}, z_{(2200)}, z_{(1120)}, z_{(1102)}, z_{(0400)}, z_{(0040)}, z_{(0022)}, z_{(0004)} \right). \end{aligned}$$

See also (8) for $(\psi_{\boldsymbol{\beta}} : \mathcal{H})$ and $(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H})$.

Define the moment cone generated by $\mathcal{H}$ and $\mathbb{L}$ as

$$\mathbb{M}(\mathcal{H}, \mathbb{L}) \quad = \quad \left\{ \sum_{p=1}^{q} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) : \boldsymbol{x}_p \in \mathbb{L} \ (p = 1, 2, \ldots, q) \text{ and } q \in \mathbb{Z}_+ \right\}. \tag{12}$$

$\mathbb{M}(\mathcal{H}, \mathbb{L})$ forms a convex cone by the following lemma. Hence, by Carathéodory's Theorem [7], the nonnegative integer q to which the summation is taken in the description of $\mathbb{M}(\mathcal{H}, \mathbb{L})$ can be fixed to $q^* = |\mathcal{H}|$;

$$\mathbb{M}(\mathcal{H}, \mathbb{L}) \quad = \quad \left\{ \sum_{p=1}^{q^*} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) : \boldsymbol{x}_p \in \mathbb{L} \ (p = 1, 2, \ldots, q^*) \right\}.$$

Lemma 3.1 *Suppose that $\mathbb{L}$ is a closed cone in $\mathbb{R}^n$ and that $\mathcal{H}_{\min} \subset \mathcal{H} \subset \mathcal{H}_{max}$.*

(a) $\mathbb{M}(\mathcal{H}, \mathbb{L})$ *is a convex cone.*

(b) Assume that $\{\tau\boldsymbol{e}_1,\ldots,\tau\boldsymbol{e}_n\}\subset\mathcal{H}$. If τ is even or $\mathbb{L}=\mathbb{R}^n_+$, then $\mathbb{M}(\mathcal{H},\mathbb{L})$ is closed, where $\boldsymbol{e}_i$ denotes the i-th coordinate unit vector of $\mathbb{R}^n$.

Proof: See Sections 4.1 and 4.2. ∎

If the assumption in (b) is not satisfied, $\mathbb{M}(\mathcal{H},\mathbb{L})$ is not necessarily closed. For example, let $n=2,\ \mathbb{L}=\mathbb{R}^2_+,\ \tau=2,\ \mathcal{H}=\{(2,0),\ (1,1)\}\not\ni(0,2)$. Then

$$\mathbb{M}(\mathcal{H},\mathbb{R}^2_+)=\left\{(x_1^2,x_1x_2)+(y_1^2,y_1y_2):\boldsymbol{x}=(x_1,x_2),\ \boldsymbol{y}=(y_1,y_2)\in\mathbb{R}^2_+\right\}.$$

If we take a sequence $\left\{\boldsymbol{x}_r=(1/r,r)\in\mathbb{R}^2_+ : r=1,2,\ldots\right\}$, then the sequence

$$\left\{((\boldsymbol{x}_r)^{\boldsymbol{\beta}}:\mathcal{H})=((1/r)^2,1)\in\mathbb{M}(\mathcal{H},\mathbb{R}^2_+):r=1,2,\ldots,\right\}$$

converges to $(0,1)\not\in\mathbb{M}(\mathcal{H},\mathbb{R}^2_+)$.

Define

$$\widehat{T}(\mathcal{H}) = \left\{(z_{\boldsymbol{\beta}}:\mathcal{H})\in\mathbb{R}^{\mathcal{H}}: \begin{array}{l}(z_{\boldsymbol{\beta}}:\mathcal{H})\in\mathbb{M}(\mathcal{H},\mathbb{L}),\\ ((h_0)_{\boldsymbol{\beta}}:\mathcal{H})\cdot(z_{\boldsymbol{\beta}}:\mathcal{H})=1,\\ ((h_j)_{\boldsymbol{\beta}}:\mathcal{H})\cdot(z_{\boldsymbol{\beta}}:\mathcal{H})=0\ (j\in J)\end{array}\right\}.$$

We introduce the moment cone relaxation of POP (1).

$$\text{minimize}\quad (\psi_{\boldsymbol{\beta}}:\mathcal{H})\cdot(z_{\boldsymbol{\beta}}:\mathcal{H})\quad\text{subject to}\quad (z_{\boldsymbol{\beta}}:\mathcal{H})\in\widehat{T}(\mathcal{H}). \tag{13}$$

Recall that $\mathcal{H}$ can be an arbitrary subset of $\mathbb{Z}^n_+$ satisfying

$$\mathcal{H}_{\min}=\text{supp}(\psi)\bigcup\left(\bigcup_{j\in J_0}\text{supp}(h_j)\right)\subset\mathcal{H}\subset\mathcal{H}_{max},\ \text{or}$$
$$\mathcal{H}_{\min}\bigcup\{\tau\boldsymbol{e}_1,\ldots,\tau\boldsymbol{e}_n\}\subset\mathcal{H}\subset\mathcal{H}_{max},$$

for the closedness of $\mathbb{M}(\mathcal{H},\mathbb{L})$ when $\mathbb{K}=\mathbb{R}^n_+$. If the polynomials $\psi,\ h_j\in\mathbb{R}[\boldsymbol{x}]\ (j\in J_0)$ of POP (1) are sparse or they involve a small number of monomials, the dimension $|\mathcal{H}|$ of the variable vector $(z_{\boldsymbol{\beta}}:\mathcal{H})$ of the problem (13) can be small. Thus, the moment cone relaxation (13) naturally inherits such sparsity from POP (1).

We note that the problems (11) and (13) have the same linear objective function $(\psi_{\boldsymbol{\beta}}:\mathcal{H})\cdot(z_{\boldsymbol{\beta}}:\mathcal{H})$. Let

$$\begin{aligned}T_0 &= \{\boldsymbol{x}\in\mathbb{L}:h_0(\boldsymbol{x})\geq 0\},\\ T_j &= \{\boldsymbol{x}\in T_{j-1}:h_j(\boldsymbol{x})=0\}\\ &= \{\boldsymbol{x}\in\mathbb{L}:h_0(\boldsymbol{x})\geq 0,\ h_i(\boldsymbol{x})=0\ (i=1,\ldots,j)\}\ \ (j\in J).\end{aligned}$$

We consider the following conditions to ensure that (11) and (13) have equivalent feasible regions in the sense that closure conv $\widetilde{T}(\mathcal{H})=$ closure $\widehat{T}(\mathcal{H})$.

$$h_0(\boldsymbol{x}) \geq 0 \text{ for every } \boldsymbol{x}\in\mathbb{L},\ i.e.,\ T_0=\mathbb{L}, \tag{14}$$
$$h_j(\boldsymbol{x}) \geq 0 \text{ for every } \boldsymbol{x}\in T_{j-1}\ (j\in J), \tag{15}$$
$$T^{\infty}_* \supset \{\boldsymbol{x}\in\mathbb{L}:h_j(\boldsymbol{x})=0\ (j\in J_0)\}. \tag{16}$$

Here, for every $A \subset \mathbb{R}^n$, A^∞ denotes the horizontal cone defined by

$$A^\infty = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \begin{array}{l} \text{there exists } (\mu_r, \boldsymbol{y}_r) \in \mathbb{R}_+ \times A \ \ (\text{r=1,2,}\ldots) \\ \text{such that } (\mu_r, \mu_r \boldsymbol{y}_r) \to (0, \boldsymbol{x}) \text{ as } r \to \infty \end{array} \right\}$$

(see, for examples, [19]). If we $\tau = 2$ or (1) represents a homogeneous QOP, conditions (14), (15) and (16) are equivalent to the set of conditions (A)', ($\widetilde{B}$), ($\widetilde{C}$) and (D) assumed in [1]. These conditions will be compared to the conditions imposed in [16] on the nonhomogeneous POP of the form (5) for the equivalence to its canonical convixification in Section 5. It is easily verified that the converse inclusion of (16) always holds.

The following theorem asserts that the closures of feasible regions conv $\widetilde{T}(\mathcal{H})$ of POP (11) and $\widehat{T}(\mathcal{H})$ of the moment cone relaxation problem (13) coincide with each other. Consequently, (11) and (13) are equivalent. Since POPs (1) and (11) have a same optimal value, (13) attains the exact optimal value of (1).

Theorem 3.2 *Assume that $\mathbb{L}$ is a closed cone and that conditions (9), (14), (15) and (16) hold. If $\mathcal{H}_{min} \subset \mathcal{H} \subset \mathcal{H}_{\max}$, then closure conv $\widetilde{T}(\mathcal{H})$ = closure $\widehat{T}(\mathcal{H})$ and*

$$\inf\left\{\psi(\boldsymbol{x}) : \boldsymbol{x} \in T_*\right\} = \inf\left\{(\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) : (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widehat{T}(\mathcal{H})\right\} \tag{17}$$

Proof: See Section 4.3 and 4.4. ∎

We conclude this section by extending the previous discussions to more general cases. This is utilized in the discussion of a doubly nonnegative cone relaxation for POP (1) in Section 6.

Corollary 3.3 *Assume that $\mathbb{L}$ is a closed cone and that conditions (9), (14), (15) and (16) hold. If $\mathcal{H}_{\min} \subset \mathcal{H}$ (but not necessarily $\mathcal{H} \subset \mathcal{H}_{\max}$), then (17) holds.*

Proof: We first observe that conditions (9), (14), (15) and (16) do not depend on any choice of $\mathcal{H} \supset \mathcal{H}_{\min}$, and that all definitions of $(\psi_{\boldsymbol{\beta}} : \mathcal{H})$, $((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}}$ $(j \in J_0)$, $\mathbb{M}(\mathcal{H}, \mathbb{L})$, $\widetilde{T}(\mathcal{H})$, and $\widehat{T}(\mathcal{H})$ remain consistent, although Lemma 3.1 may not be true. We can easily verify that $\widetilde{T}(\mathcal{H}) \subset \widehat{T}(\mathcal{H})$. Hence,

$$\begin{aligned} \inf\left\{\psi(\boldsymbol{x}) : \boldsymbol{x} \in T_*\right\} &= \inf\left\{(\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) : (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widetilde{T}(\mathcal{H})\right\} \\ &\geq \inf\left\{(\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) : (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widehat{T}(\mathcal{H})\right\}. \end{aligned}$$

On the other hand, if $(\bar{z}_{\boldsymbol{\beta}} : \mathcal{H}) \in \widehat{T}(\mathcal{H})$, then $(\bar{z}_{\boldsymbol{\beta}} : \mathcal{H} \bigcap \mathcal{H}_{\max}) \in \widehat{T}(\mathcal{H} \bigcap \mathcal{H}_{\max}))$ and $(\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\bar{z}_{\boldsymbol{\beta}} : \mathcal{H}) = (\psi_{\boldsymbol{\beta}} : \mathcal{H} \bigcap \mathcal{H}_{\max}) \cdot (\bar{z}_{\boldsymbol{\beta}} : \mathcal{H} \bigcap \mathcal{H}_{\max})$. Therefore,

$$\begin{aligned} & \inf\left\{(\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) : \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widehat{T}(\mathcal{H})\right\} \\ \geq \ & \inf\Big\{(\psi_{\boldsymbol{\beta}} : \mathcal{H} \bigcap \mathcal{H}_{\max}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H} \bigcap \mathcal{H}_{\max}) : \\ & \qquad\qquad\qquad (z_{\boldsymbol{\beta}} : \mathcal{H} \bigcap \mathcal{H}_{\max}) \in \widehat{T}(\mathcal{H} \bigcap \mathcal{H}_{\max})\Big\} \\ = \ & \inf\left\{\psi(\boldsymbol{x}) : \boldsymbol{x} \in T_*\right\}. \end{aligned}$$

Here the last equality follows from Theorem 3.2. ∎

4 Proof

4.1 Proof of (a) in Lemma 3.1

Suppose that $\sum_{p=1}^{q}((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{M}(\mathcal{H},\mathbb{L})$, $\boldsymbol{x}_p \in \mathbb{L}$ $(p = 1,\ldots,q)$, $\sum_{p=1}^{\bar{q}}(\bar{\boldsymbol{x}}_p^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{M}(\mathcal{H},\mathbb{L})$, $\bar{\boldsymbol{x}}_p \in \mathbb{L}$ $(p = 1,\ldots,\bar{q})$, $\lambda \geq 0$ and $\bar{\lambda} \geq 0$. Since $\mathbb{L}$ is a cone, we see that $\lambda^{1/\tau}\boldsymbol{x}_p \in \mathbb{L}$ $(p = 1,\ldots,q)$ and $\bar{\lambda}^{1/\tau}\bar{\boldsymbol{x}}_p \in \mathbb{L}$ $(p = 1,\ldots,\bar{q})$. By $\mathcal{H} \subset \mathcal{H}_{\max} = \left\{\boldsymbol{\beta} \in \mathbb{Z}^n_+ : |\boldsymbol{\beta}|_1 = \tau\right\}$,

$$
\begin{aligned}
&\lambda\sum_{p=1}^{q}((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) + \bar{\lambda}\sum_{p=1}^{\bar{q}}((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \\
&\quad = \sum_{p=1}^{q}((\lambda^{1/\tau}\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p=1}^{\bar{q}}((\bar{\lambda}^{1/\tau}\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{M}(\mathcal{H},\mathbb{L}).
\end{aligned}
$$

Thus we have shown that $\mathbb{M}(\mathcal{H},\mathbb{L})$ is a convex cone.

4.2 Proof of (b) in Lemma 3.1

Consider a sequence

$$
\begin{aligned}
\mathbb{M}(\mathcal{H},\mathbb{L}) \ni ((z_r)_{\boldsymbol{\beta}} : \mathcal{H}) &= \sum_{p=1}^{q}((\boldsymbol{x}_{rp})^{\boldsymbol{\beta}} : \mathcal{H}) \\
&\quad \text{with } \boldsymbol{x}_{rp} \in \mathbb{L} \ (p = 1,2,\ldots,q) \ (r = 1,2,\ldots\ ), \qquad (18)
\end{aligned}
$$

which converges to some $(\bar{z}_{\boldsymbol{\beta}} : \mathcal{H})$ as $r \to \infty$. We show that the sequence $\{\boldsymbol{x}_{rp} \in \mathbb{L} : r = 1,2,\ldots\ \}$ is bounded $(p = 1,2,\ldots,q)$. From (18), we observe that

$$
\sum_{p=1}^{q}(\boldsymbol{x}_{rp})^{\boldsymbol{\beta}} = (z_r)_{\boldsymbol{\beta}} \to \bar{z}_{\boldsymbol{\beta}} \ \text{ as } r \to \infty \ (\boldsymbol{\beta} \in \mathcal{H}).
$$

By the assumption, we know that $\{\tau\boldsymbol{e}_1,\ldots,\tau\boldsymbol{e}_n\} \subset \mathcal{H}$. As a result, the above relation holds for $\boldsymbol{\beta} = \tau\boldsymbol{e}_i \in \mathcal{H}$ $(i = 1,\ldots,n)$. If each $\boldsymbol{x}_{rp}$ is denoted as $(x_{rp1},\ldots,x_{rpn})$, then $(\boldsymbol{x}_{rp})^{(\tau\boldsymbol{e}_i)} = (x_{rpi})^\tau \geq 0$ since τ is a even integer or $\mathbb{L} = \mathbb{R}^n_+$ by the assumption. Hence, we obtain that

$$
\begin{aligned}
0 \leq (x_{rpi})^\tau \leq \sum_{q=1}^{q}(x_{rqi})^\tau = \sum_{q=1}^{q}(\boldsymbol{x}_{rq})^{(\tau\boldsymbol{e}_i)} = (z_r)_{(\tau\boldsymbol{e}_i)} \to \bar{z}_{(\tau\boldsymbol{e}_i)} \ \text{ as } r \to \infty \\
\text{for } i = 1,\ldots,n \text{ and } p = 1,2,\ldots,q \ (r = 1,2,\ldots,\ ). \qquad (19)
\end{aligned}
$$

This implies that all sequences $\{\boldsymbol{x}_{rp} \in \mathbb{L} : r = 1,2,\ldots,\ \}$ $(p = 1,2,\ldots,q)$ are bounded. Thus, we can take a subsequence of (18) along which $\boldsymbol{x}_{rp} \in \mathbb{L}$ converges to some $\bar{\boldsymbol{x}}_p \in \mathbb{L}$ as $r \to \infty$ $(p = 1,2,\ldots,q)$. Therefore, $(\bar{z}^{\boldsymbol{\beta}} : \mathcal{H}) = \sum_{p=1}^{q}((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{M}(\mathcal{H},\mathbb{L})$.

4.3 Proof of closure conv $\widetilde{T}(\mathcal{H}) \subset$ closure $\widehat{T}(\mathcal{H})$ in Theorem 3.2

Assume that $(z_{\boldsymbol{\beta}} : \mathcal{H}) = (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}) \in \widetilde{T}(\mathcal{H})$. Then, $(z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widehat{T}(\mathcal{H})$ by definition. Since closure $\widehat{T}(\mathcal{H})$ is convex and closed, closure conv $\widetilde{T}(\mathcal{H}) \subseteq$ closure $\widehat{T}(\mathcal{H})$ follows.

4.4 Proof of closure $\widehat{T}(\mathcal{H}) \subset$ closure conv $\widetilde{T}(\mathcal{H})$ in Theorem 3.2

It suffices to show that $\widehat{T}(\mathcal{H}) \subset$ closure conv $\widetilde{T}(\mathcal{H})$. Suppose that $(z_{\boldsymbol{\beta}} : \mathcal{H}) \in \widehat{T}(\mathcal{H})$. Then,

$$((h_0)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) = 1, \quad ((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) = 0 \ (j \in J),$$
$$(z_{\boldsymbol{\beta}} : \mathcal{H}) = \sum_{p=1}^{q} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \ \text{ for some } \boldsymbol{x}_p \in \mathbb{L} \ (p = 1, \ldots, q).$$

It follows that

$$\begin{aligned}
1 &= ((h_0)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot \left(\sum_{p=1}^{q} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \right) \\
&= \sum_{p=1}^{q} \big(((h_0)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \big) = \sum_{p=1}^{q} h_0(\boldsymbol{x}_p), \\
0 &= ((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot \left(\sum_{p=1}^{q} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \right) \\
&= \sum_{p=1}^{q} \big(((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) \cdot ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \big) = \sum_{p=1}^{q} h_j(\boldsymbol{x}_p) \ (j \in J = \{1, \ldots, \ell\}). \qquad (20)
\end{aligned}$$

We will show by induction that

$$\boldsymbol{x}_p \in T_j \ (p = 1, \ldots, q) \ (j = 0, \ldots, \ell). \qquad (21)$$

It follows from $\boldsymbol{x}_p \in \mathbb{L}$ and (14) that $\boldsymbol{x}_p \in T_0$. Now assume that $\boldsymbol{x}_p \in T_{j-1}$ for some j with $j \in J$ $(p = 1, \ldots, q)$. By (15), we see that $h_j(\boldsymbol{x}_p) \geq 0$ $(p = 1, \ldots, q)$. Hence (20) implies that $h_j(\boldsymbol{x}_p) = 0$, and $\boldsymbol{x}_p \in T_j$ $(p = 1, \ldots, q)$. Thus we have shown (21).

From $\boldsymbol{x}_p \in T_0$, we know that $\lambda_p = h_0(\boldsymbol{x}_p)$ is nonnegative $(p = 1, 2, \ldots, q)$. Let

$$\begin{aligned}
I_+ &= \{p : \lambda_p = h_0(\boldsymbol{x}_p) > 0\}, \ I_0 = \{p : \lambda_p = h_0(\boldsymbol{x}_p) = 0\}, \\
\bar{\boldsymbol{x}}_p &= \boldsymbol{x}_p / (\lambda_p)^{1/\tau} \in \mathbb{L} \ (p \in I_+).
\end{aligned}$$

By (9), (10) and $\mathcal{H}_{\min} \subset \mathcal{H} \subset \mathcal{H}_{\max}$ that

$$\begin{aligned}
((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) &= (((\lambda_p)^{1/\tau} \bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) = \lambda_p ((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \ (p \in I_+), \\
h_0(\bar{\boldsymbol{x}}_p) &= h_0(\boldsymbol{x}_p / \lambda_p^{1/\tau}) = h_0(\boldsymbol{x}_p)/\lambda_p = 1 \ (p \in I_+), \\
h_j(\bar{\boldsymbol{x}}_p) &= h_j(\boldsymbol{x}_p / \lambda_p^{1/\tau}) = h_j(\boldsymbol{x}_p)/\lambda_p = 0 \ \ (p \in I_+) \ (j \in J).
\end{aligned}$$

Hence

$$\begin{aligned}
((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) &\in \widetilde{T}(\mathcal{H}) \ \ (p \in I_+), \\
1 &= \sum_{p=1}^{q} h_0(\boldsymbol{x}_p) = \sum_{p \in I_+} h_0(\boldsymbol{x}_p) = \sum_{p \in I_+} \lambda_p, \ \lambda_p > 0 \ (p \in I_+), \\
(z_{\boldsymbol{\beta}} : \mathcal{H}) &= \sum_{p=1}^{q} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \ = \sum_{p \in I_+} \lambda_p ((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p \in I_0} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}), \\
\boldsymbol{x}_p &\in \{\boldsymbol{x} \in \mathbb{L} : h_j(\boldsymbol{x}) = 0 \ (j \in J_0)\} \ \ (p \in I_0).
\end{aligned}$$

By (16), for each $p \in I_0$, there exists a sequence $\left\{(\mu_{pr}, \boldsymbol{y}_{pr}) \in \mathbb{R}_+ \times T_*\right\}$ such that $(\mu_{pr}, \mu_{pr}\boldsymbol{y}_{pr}) \to (0, \boldsymbol{x}_p)$ as $r \to \infty$. Let $\tilde{p} \in I_+$ and $\tilde{I}_+ = I_+ \backslash \{\tilde{p}\}$. Then, for sufficiently large r such that $\lambda_{\tilde{p}} - \sum_{p \in I_0} (\mu_{rp})^\tau > 0$,

$$
\begin{aligned}
&\text{conv } \widetilde{T}(\mathcal{H}) \\
\ni \quad & \left(\lambda_{\tilde{p}} - \sum_{p \in I_0} (\mu_{rp})^\tau\right) ((\bar{\boldsymbol{x}}_{\tilde{p}})^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p \in \tilde{I}_+} \lambda_p ((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p \in I_0} (\mu_{rp})^\tau ((\boldsymbol{y}_{pr})^{\boldsymbol{\beta}} : \mathcal{H}) \\
= \quad & \left(\lambda_{\tilde{p}} - \sum_{p \in I_0} (\mu_{rp})^p\right) ((\bar{\boldsymbol{x}}_{\tilde{p}})^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p \in \tilde{I}_+} \lambda_p ((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p \in I_0} ((\mu_{rp} \boldsymbol{y}_{pr})^{\boldsymbol{\beta}} : \mathcal{H}) \\
\to \quad & \sum_{p \in I_+} \lambda_p ((\bar{\boldsymbol{x}}_p)^{\boldsymbol{\beta}} : \mathcal{H}) + \sum_{p \in I_0} ((\boldsymbol{x}_p)^{\boldsymbol{\beta}} : \mathcal{H}) = (z_{\boldsymbol{\beta}} : \mathcal{H}) \text{ as } r \to \infty.
\end{aligned}
$$

Therefore we have shown that $(z_{\boldsymbol{\beta}} : \mathcal{H}) \in$ closure conv $\widetilde{T}(\mathcal{H})$.

5 Nonhomogeneous model

The discussions up to this point has been focused on POP (1) described by homogeneous polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}]$ $(j \in J_0)$ characterized by condition (2). In this section, we deal with POP of the form (5) described by general (nonhomogeneous) polynomials $\varphi,\ g_j \in \mathbb{R}[\boldsymbol{w}]$ $(j \in J = \{1, \ldots, \ell\})$ with any degrees, where $\boldsymbol{w} = (w_1, \ldots, w_m) \in \mathbb{R}^m$. Peña, Vera and Zuluaga [16] applied their canonical convexification procedure to this type of POP (5) with $\mathbb{K} = \mathbb{R}^m_+$, and presented a linear optimization problem over the cone of completely positive d-forms equivalent to POP (5) in Theorem 10 of [16]. We impose conditions similar to but weaker than theirs, and convert POP (5) into POP (1) satisfying conditions (9), (14), (15) and (16); hence Theorem 3.2 holds.

Let

$$
\begin{aligned}
\tau &= \max\{\deg(\varphi),\ \deg(g_j)\ (j \in J)\}, \\
\mathcal{G}_{\min} &= \text{supp}(\varphi) \bigcup \left(\bigcup_{j \in J} \text{supp}(g_j)\right),\ \mathcal{G}_{\max} = \left\{\boldsymbol{\alpha} \in \mathbb{Z}^m_+ : |\boldsymbol{\alpha}|_1 \leq \tau\right\}.
\end{aligned}
$$

Choose $\mathcal{G} \subset \mathbb{Z}^m_+$ such that $\mathcal{G}_{\min} \bigcup \{\boldsymbol{0}\} \subset \mathcal{G} \subset \mathcal{G}_{\max}$. Then, $\varphi,\ g_j \in \mathbb{R}[\boldsymbol{w}]$ $(j \in J)$ can be represented as

$$
\begin{aligned}
\varphi(\boldsymbol{w}) &= (\varphi_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (\boldsymbol{w}^{\boldsymbol{\alpha}} : \mathcal{G}) \text{ for some } (\varphi_{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathbb{R}^{\mathcal{G}}, \\
g_j(\boldsymbol{w}) &= ((g_j)_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (\boldsymbol{w}^{\boldsymbol{\alpha}} : \mathcal{G}) \text{ for some } ((g_j)_{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathbb{R}^{\mathcal{G}}\ (j \in J)
\end{aligned}
$$

Let $\text{Copos}(\mathcal{G}, \mathbb{K})^* = \text{cone conv } \left\{(\boldsymbol{w}^{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathbb{R}^{\mathcal{G}} : \boldsymbol{w} \in \mathbb{K}\right\}$. We now consider the linear conic program over the cone $\text{Copos}(\mathcal{G}, \mathbb{K})^*$

$$
\text{minimize} \quad (\varphi_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (y_{\boldsymbol{\alpha}} : \mathcal{G}) \quad \text{subject to} \quad (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \widehat{S}(\mathcal{G}), \tag{22}
$$

where

$$
\begin{aligned}
\widehat{S}(\mathcal{G}) &= \left\{ (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathbb{R}^{\mathcal{G}} : \begin{array}{l} (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathrm{Copos}(\mathcal{G}, \mathbb{K})^*, \\ ((g_0)_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (y_{\boldsymbol{\alpha}} : \mathcal{G}) = 1, \\ ((g_j)_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (y_{\boldsymbol{\alpha}} : \mathcal{G}) = 0 \ (j \in J) \end{array} \right\}, \\
g_0(\boldsymbol{w}) &= ((g_0)_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (\boldsymbol{w}^{\boldsymbol{\alpha}} : \mathcal{G}), \text{ where } (g_0)_{\boldsymbol{\alpha}} = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} = \mathbf{0} \in \mathcal{G}, \\ 0 & \text{if } \boldsymbol{\alpha} \in \mathcal{G} \text{ and } \boldsymbol{\alpha} \neq \mathbf{0}, \end{cases} .
\end{aligned}
$$

We note that $g_0 \in \mathbb{R}[\boldsymbol{w}, \mathcal{G}]$ has been consistently defined since $\mathbf{0} \in \mathcal{G}$. In the literature [16], the cone $\mathrm{Copos}(\mathcal{G}, \mathbb{K})^*$ is called the cone of completely positive d-forms and the procedure deriving the problem (22) from POP (5) the canonical convexification procedure.

By construction, if $\boldsymbol{w} \in \mathbb{R}^m$ is a feasible solution of POP (5), then $(y_{\boldsymbol{\alpha}} : \mathcal{G}) = (\boldsymbol{w}^{\boldsymbol{\alpha}} : \mathcal{G})$ is a feasible solution of the problem (22), and the objective value $(\varphi_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (y_{\boldsymbol{\alpha}} : \mathcal{G})$ coincides with the objective value $\varphi(\boldsymbol{w})$ at $\boldsymbol{w} \in \mathbb{R}^m$. Therefore, the problem (22) serves as a relaxation problem of POP (5), and

$$
\inf \{ \varphi(\boldsymbol{w}) : \boldsymbol{w} \in S_* \} \geq \inf \left\{ (\varphi_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (y_{\boldsymbol{\alpha}} : \mathcal{G}) : (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \widehat{S}(\mathcal{G}) \right\}, \tag{23}
$$

where S_* denotes the feasible region of POP (5).

It is interesting to note that $\mathrm{Copos}(\mathcal{G}, \mathbb{K})^*$ is not closed if we take $\mathbb{K} = \mathbb{R}^m_+$ and $\mathcal{G} = \mathcal{G}_{\max}$ as in [16]. In fact, the following stronger assertion is true.

Lemma 5.1 *Suppose that $\tau \geq 1$ and $\boldsymbol{d}^{\boldsymbol{\gamma}} \neq 0$ for some $\boldsymbol{d} \in \mathbb{K}$ and some $\boldsymbol{\gamma} \in \mathcal{G}$ with $|\boldsymbol{\gamma}|_1 = \tau$. Then $\mathrm{Copos}(\mathcal{G}, \mathbb{K})^*$ is not closed. (We have assumed that $\mathbf{0} \in \mathcal{G}$).*

Proof: Let $\boldsymbol{x}(\mu) = \mu \boldsymbol{d} \in \mathbb{K}$ for every $\mu > 0$. Then, we see that

$$
\lim_{\mu \to \infty} \frac{\boldsymbol{x}(\mu)^{\boldsymbol{\alpha}}}{\mu^\tau} = \begin{cases} \boldsymbol{d}^{\boldsymbol{\alpha}} & \text{if } \boldsymbol{\alpha} \in \mathcal{G} \text{ and } |\boldsymbol{\alpha}|_1 = \tau, \\ 0 & \text{if } \boldsymbol{\alpha} \in \mathcal{G} \text{ and } |\boldsymbol{\alpha}|_1 < \tau. \end{cases}
$$

Hence $(1/\mu^\tau)(\boldsymbol{x}(\mu)^{\boldsymbol{\alpha}} : \mathcal{G})$ converges to some nonzero $(\bar{y}_{\boldsymbol{\alpha}} : \mathcal{G}) \in$ closure $\mathrm{Copos}(\mathcal{G}, \mathbb{K})^*$ such that $\bar{y}_{\mathbf{0}} = 0$ and $\bar{y}_{\boldsymbol{\gamma}} = \boldsymbol{d}^{\boldsymbol{\gamma}} \neq 0$. On the other hand, we know that the cone $\mathrm{Copos}(\mathcal{G}, \mathbb{K})^*$ is included in the halfspace $\left\{ (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathbb{R}^{\mathcal{G}} : y_{\mathbf{0}} \geq 0 \right\}$, and that $(\bar{y}_{\boldsymbol{\alpha}} : \mathcal{G})$ lies in its facet, the hyperplane $\left\{ (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \mathbb{R}^{\mathcal{G}} : y_{\mathbf{0}} = 0 \right\}$. Hence, if $\mathbf{0} \neq (\bar{y}_{\boldsymbol{\alpha}} : \mathcal{G}) = \sum_{p=1}^q \lambda_p ((\boldsymbol{w}_p)^{\boldsymbol{\alpha}} : \mathcal{G})$ for some $\lambda_p > 0$, $\boldsymbol{w}_p \in \mathbb{R}^m_+$ $(p = 1, \ldots, q)$ and some $q \geq 1$ holds, then $((\boldsymbol{w}_p)^{\boldsymbol{\alpha}} : \mathcal{G})$ must lie in the halfspace. This is impossible because $(\boldsymbol{w}_p)^{\mathbf{0}} = 1$ $(p = 1, \ldots, q)$. ∎

We now convert POP (5) into POP (1), and the problem (22) into the moment cone problem (13), respectively, then, show the identity (17) by applying Theorem 3.2. Let $n = 1 + m$, $\mathbb{L} = \mathbb{R}_+ \times \mathbb{K}$ and $J_0 = \{0\} \bigcup J$. Define $\theta : \mathcal{G} \to \mathbb{Z}^n_+$ by $\theta(\boldsymbol{\alpha}) = (\tau - |\boldsymbol{\alpha}|_1, \boldsymbol{\alpha})$ for every $\boldsymbol{\alpha} \in \mathcal{G}$. It is obvious that θ is one-to-one mapping from $\mathcal{G}$ onto its image $\mathcal{H} = \theta(\mathcal{G}) = \{\theta(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathcal{G}\}$. Thus, the $|\mathcal{G}|$-dimensional space $\mathbb{R}^{\mathcal{G}}$ can be identified with the $|\mathcal{H}|$-dimensional space $\mathbb{R}^{\mathcal{H}}$; the coordinate index $\boldsymbol{\alpha} \in \mathcal{G}$ of the space $\mathbb{R}^{\mathcal{G}}$ corresponds to the coordinate index $\theta(\boldsymbol{\alpha}) \in \mathcal{H}$ of the space $\mathbb{R}^{\mathcal{H}}$ and vice vera. Specifically, the coordinate index $\mathbf{0} \in \mathcal{G}$ corresponds to $\theta(\mathbf{0}) = (\tau, \mathbf{0}) \in \mathcal{H}$. As a result, the polynomials ψ, $h_j \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ $(j \in J_0)$ can be consistently defined by

$$
\begin{aligned}
\psi(\boldsymbol{x}) &= (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}), \text{where } (\psi_{\boldsymbol{\beta}} : \mathcal{H}) = (\varphi_{\theta(\boldsymbol{\alpha})} : \mathcal{G}) \in \mathbb{R}^{\mathcal{H}}, \\
h_j(\boldsymbol{x}) &= (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{H}), \text{where } ((h_j)_{\boldsymbol{\beta}} : \mathcal{H}) = ((g_j)_{\theta(\boldsymbol{\alpha})} : \mathcal{G}) \in \mathbb{R}^{\mathcal{H}} \ (j \in J_0).
\end{aligned}
$$

We observe that, by construction,

$$\begin{array}{rcl} h_0(\boldsymbol{x}) & = & (w_0)^\tau \text{ for every } \boldsymbol{x} = (w_0, \boldsymbol{w}) \in \mathbb{L}, \\ \psi(\boldsymbol{x}) & = & \varphi(\boldsymbol{w}),\ h_j(\boldsymbol{x}) = g_j(\boldsymbol{w})\ (j \in J) \\ & & \qquad\qquad \text{if } \boldsymbol{x} = (w_0, \boldsymbol{w}) \in \mathbb{L} \text{ satisfies } h_0(\boldsymbol{x}) = w_0^\tau = 1. \end{array} \tag{24}$$

Therefore, POP (5) is equivalent to POP (1) with these polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ $(j \in J_0)$ and the cone $\mathbb{L} = \mathbb{R}_+ \times \mathbb{K}$. Thus,

$$\inf \{\varphi(\boldsymbol{w}) : \boldsymbol{w} \in S_*\} = \inf \{\psi(\boldsymbol{x}) : \boldsymbol{x} \in T_*\}. \tag{25}$$

Now define

$$\mathbb{M}^o(\mathcal{H}, \mathbb{L}) = \left\{ \sum_{p=1}^{q^*} ((w_{p0}, \boldsymbol{w}_p)^{\boldsymbol{\beta}} : \mathcal{H}) : \begin{array}{l} w_{p0} > 0,\ (w_{p0}, \boldsymbol{w}_p) \in \mathbb{L} \\ (p = 1, \ldots, q),\ q \geq 0 \end{array} \right\}.$$

Lemma 5.2 *Copos*$(\mathcal{G}, \mathbb{K})^* = \mathbb{M}^o(\mathcal{H}, \mathbb{L}) \subset \mathbb{M}(\mathcal{H}, \mathbb{L})$.

Proof: Suppose that $w_{p0} > 0$ and $(w_{p0}, \boldsymbol{w}_p) \in \mathbb{L}$ $(p = 1, \ldots, q)$. Then

$$\begin{array}{rcl} \displaystyle\sum_{p=1}^{q} ((w_{p0}, \boldsymbol{w}_p)^{\boldsymbol{\beta}} : \mathcal{H}) & = & \displaystyle\sum_{p=1}^{q} (w_{p0})^\tau ((1, \boldsymbol{w}_p / w_{p0})^{\boldsymbol{\beta}} : \mathcal{H}) \\ & = & \displaystyle\sum_{p=1}^{q} (w_{p0})^\tau ((\boldsymbol{w}_p / w_{p0})^{\boldsymbol{\alpha}} : \mathcal{G}) \in \text{Copos}(\mathcal{G}, \mathbb{K})^*. \end{array}$$

Now suppose that $(y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \text{Copos}(\mathcal{G}, \mathbb{K})^*$. Then there exist $\lambda_p > 0$ and $\boldsymbol{w}_p \in \mathbb{K}$ $(p = 1, \ldots, q)$ such that $(y_{\boldsymbol{\alpha}} : \mathcal{G}) = \sum_{p=1}^{q} \lambda_p ((\boldsymbol{w}_p)^{\boldsymbol{\alpha}} : \mathcal{G})$. Hence,

$$\begin{array}{rcl} (y_{\boldsymbol{\alpha}} : \mathcal{G}) & = & \displaystyle\sum_{p=1}^{q} \lambda_p ((1, \boldsymbol{w}_p)^{\boldsymbol{\beta}} : \mathcal{H}) = \sum_{p=1}^{q} (((\lambda_p)^{1/\tau}, (\lambda_p)^{1/\tau} \boldsymbol{w}_p)^{\boldsymbol{\beta}} : \mathcal{H}) \\ & \in & \left\{ \displaystyle\sum_{p=1}^{q} ((w_{p0}, \boldsymbol{w}_p)^{\boldsymbol{\beta}} : \mathcal{H}) : w_{p0} > 0,\ (w_{p0}, \boldsymbol{w}_p) \in \mathbb{L}\ (p = 1, \ldots, q) \right\}. \end{array}$$

Thus, we have shown the desired identity. The latter inclusion relation follows directly from definition. ∎

By Lemma 5.2, we can rewrite the problem (22) as

$$\text{minimize} \quad (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) \quad \text{subject to} \quad (z_\beta : \mathcal{H}) \in \widehat{T}^o(\mathcal{H}), \tag{26}$$

where

$$\widehat{T}^o(\mathcal{H}) = \left\{ (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{R}^{\mathcal{H}} : \begin{array}{l} (z_{\boldsymbol{\beta}} : \mathcal{H}) \in \mathbb{M}^o(\mathcal{H}, \mathbb{L}), \\ ((h_0)_{\boldsymbol{\alpha}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) = 1, \\ ((h_j)_{\boldsymbol{\alpha}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) = 0\ (j \in J) \end{array} \right\}.$$

Since $\widehat{T}^o(\mathcal{H}) \subset \widehat{T}(\mathcal{H})$, we obtain that

$$\begin{aligned} & \inf \left\{ (\varphi_{\boldsymbol{\alpha}} : \mathcal{G}) \cdot (y_{\boldsymbol{\alpha}} : \mathcal{G}) : (y_{\boldsymbol{\alpha}} : \mathcal{G}) \in \widehat{S}(\mathcal{G}) \right\} \\ = \ & \inf \left\{ (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) : (z_\beta : \mathcal{H}) \in \widehat{T}^o(\mathcal{H}) \right\} \\ \geq \ & \inf \left\{ (\psi_{\boldsymbol{\beta}} : \mathcal{H}) \cdot (z_{\boldsymbol{\beta}} : \mathcal{H}) : (z_\beta : \mathcal{H}) \in \widehat{T}(\mathcal{H}) \right\}. \end{aligned} \quad (27)$$

For the conditions imposed on POP (5), we need to introduce some notation and symbols. Let $S_j = \{\boldsymbol{w} \in \mathbb{K} : g_i(\boldsymbol{w}) = 0 \ (i < j)\} \ (j \in J)$, $\widehat{\mathcal{G}} = \{\boldsymbol{\alpha} \in \mathcal{G} : |\boldsymbol{\alpha}|_1 = \tau\}$. For each $j \in J$, the homogeneous component of g_j with degree τ is written as $\hat{g}_j(\boldsymbol{w}) = ((g_j)_{\boldsymbol{\alpha}} : \widehat{\mathcal{G}}) \cdot (\boldsymbol{w}^{\boldsymbol{\alpha}} : \widehat{\mathcal{G}})$. We assume the following conditions.

$$g_j(\boldsymbol{w}) \ \geq \ 0 \text{ for every } \boldsymbol{w} \in S_j \ (j \in J), \quad (28)$$

$$S^\infty_* \ \supset \ \{\boldsymbol{w} \in \mathbb{K} : \hat{g}_j(\boldsymbol{w}) = 0 \ (j \in J)\}. \quad (29)$$

These conditions do not depend on any choice of $\mathcal{G} \subset \mathbb{Z}^m_+$ such that $\mathcal{G}_{\min} \bigcup \{\boldsymbol{0}\} \subset \mathcal{G} \subset \mathcal{G}_{\max}$. Condition (28) is equivalent to the one assumed in Theorem 10 of [16], while Condition (29) is weaker and simpler than the one assumed there. In addition, we can take any closed (even nonconvex and/or nonpointed) cone in $\mathbb{R}^m$ in POP (5), while the cone $\mathbb{K}$ is restricted to $\mathbb{R}^m_+$ in Theorem 10 of [16].

If we define $\mathcal{H}_{\min} = \text{supp}(\psi) \bigcup \left(\bigcup_{j \in J} \text{supp}(h_j) \right)$ and $\mathcal{H}_{\max} = \left\{ \boldsymbol{\beta} \in \mathbb{Z}^n_+ : |\boldsymbol{\beta}|_1 = \tau \right\}$, then $\mathcal{H}_{\min} \subset \mathcal{H} \subset \mathcal{H}_{\max}$ obviously holds. In addition, condition (9) holds by construction. In the remaining of this section, we show that conditions (14), (15) and (16) are satisfied to apply Theorem 3.2.

By definition, $h_0(\boldsymbol{x}) = w_0^\tau$ for every $\boldsymbol{x} = (w_0, \boldsymbol{w}) \in \mathbb{L} = \mathbb{R}_+ \times \mathbb{K}$. Thus, (14) follows. Let $j \in J$. By (9), we observe that the identity

$$h_j(w_0, \boldsymbol{w}) = (w_0)^\tau h_j(1, \boldsymbol{w}/((w_0)^\tau))) = (w_0)^\tau g_j(\boldsymbol{w}/((w_0)^\tau)))$$

holds for every $\boldsymbol{x} = (w_0, \boldsymbol{w}) \in \mathbb{L}$ with $w_0 > 0$. Hence,

$$\begin{aligned} & h_j(w_0, \boldsymbol{w}) \geq \text{ or } = 0 \text{ for every } \boldsymbol{x} = (w_0, \boldsymbol{w}) \in \mathbb{L} = \mathbb{R}_+ \times \mathbb{K} \text{ with } w_0 > 0 \\ & \qquad \text{if and only if } g_j(\boldsymbol{w}) \geq \text{ or } 0 \text{ for every } \boldsymbol{w} \in \mathbb{K}, \text{ respectively.} \end{aligned}$$

By the continuity, we can relax the restriction $w_0 > 0$ into $w_0 \geq 0$, and obtain that

$$\begin{aligned} & h_j(w_0, \boldsymbol{w}) \geq \text{ or } = 0 \text{ for every } \boldsymbol{x} = (w_0, \boldsymbol{w}) \in \mathbb{L} = \mathbb{R}_+ \times \mathbb{K} \\ & \qquad \text{if and only if } g_j(\boldsymbol{w}) \geq \text{ or } 0 \text{ for every } \boldsymbol{w} \in \mathbb{K}, \text{ respectively.} \end{aligned}$$

This relation holds for every $j \in J$. Therefore, (15) follows from (28).

Assume that $\boldsymbol{x} = (w_0, \boldsymbol{w}) \in \{\boldsymbol{x} \in \mathbb{L} : h_j(\boldsymbol{x}) = 0 \ (j \in J_0)\}$. Then $\boldsymbol{w} \in \mathbb{K}$, $w_0 = 0$ and $0 = h_j(0, \boldsymbol{w}) = \hat{g}_j(\boldsymbol{w}) \ (j \in J)$. By condition (29), there exists a sequence $\{(\mu_r, \boldsymbol{v}_r) \in \mathbb{R}^n\}$ such that $(\mu_r, \boldsymbol{v}_r) \in \mathbb{R}_+ \times \mathbb{K}$, $g_j(\boldsymbol{v}_r) = 0 \ (j \in J)$ and $(\mu_r, \mu_r \boldsymbol{v}_r) \to (0, \boldsymbol{w})$ as $r \to \infty$. By letting $\boldsymbol{y}_r = (1, \boldsymbol{v}_r) \in \mathbb{L}$ (($r = 1, 2, \ldots,$), we have

$$\begin{aligned} & (\mu_r, \boldsymbol{y}_r) \in \mathbb{R}_+ \times \mathbb{L}, \ h_0(\boldsymbol{y}_r) = 1, \ h_j(\boldsymbol{y}_r) = g_j(\boldsymbol{v}_r) = 0 \ (j \in J), \\ & (\mu_r, \mu_r \boldsymbol{y}_r) = (\mu_r, (\mu_r, \mu_r \boldsymbol{v}_r)) \to (0, (0, \boldsymbol{w})) = (0, \boldsymbol{x}) \text{ as } r \to \infty. \end{aligned}$$

This implies that $\boldsymbol{x} \in T_*^\infty$. Consequently, we have shown (16).

By applying Theorem 3.2, we know that the identity (17) is satisfied. Taking account of all equalities and inequalities in (17), (23), (25) and (27), we finally conclude that the equality holds in the inequality (23), *i.e.*, POP (5) and its relaxation (22) have a same optimal objective value.

6 Doubly nonnegative cone relaxation

We assume $\mathbb{K} = \mathbb{R}^n_+$ in this section. We apply a doubly nonnegative cone relaxation to the moment cone programming problem (13) with a special choice of $\mathcal{H}$ in Section 6.1, and show how sparsity can be exploited in the doubly nonnegative cone relaxation in Section 6.2. In Section 6.3, we briefly discuss on how the idea of doubly nonnegative cone relaxation can be incorporated into Lasserre's SDP relaxation.

6.1 Relaxation of the moment cone problem

We choose an $\mathcal{H} \subset \mathbb{Z}^n_+$ such that $\mathcal{H}_{\min} \subset \mathcal{H} = \mathcal{F} + \mathcal{F}$ for some $\mathcal{F} \subset \mathbb{Z}^n_+$. In general, we can take

$$\mathcal{F} \;=\; \begin{cases} \left\{\boldsymbol{\beta} \in \mathbb{Z}^n_+ : |\boldsymbol{\beta}|_1 = \tau/2\right\} & \text{if } \tau \text{ is even,} \\ \left\{\boldsymbol{\beta} \in \mathbb{Z}^n_+ : \lfloor \tau/2 \rfloor \leq |\boldsymbol{\beta}|_1 \leq \lceil \tau/2 \rceil\right\} & \text{otherwise.} \end{cases}$$

To generate a small $\mathcal{F}$, we can apply the following heuristic algorithm.

Algorithm 6.1

Step 0: Let $k = 0$. Choose an initial $\mathcal{F} = \mathcal{F}_0 \subset \mathbb{Z}^n_+$ which satisfies $\mathcal{H}_{\min} \subset \mathcal{F} + \mathcal{F}$.

Step 1: Choose $\{\boldsymbol{\beta}\} \in \mathcal{F}_k$ such that $\mathcal{F} = \mathcal{F}_k \backslash \{\boldsymbol{\beta}\}$ satisfies $\mathcal{H}_{\min} \subset \mathcal{F} + \mathcal{F}$. If such a $\{\boldsymbol{\beta}\} \in \mathcal{F}_k$ does not exist, output $\mathcal{F} = \mathcal{F}_k$, and stop the iteration.

Step 2: Let $\mathcal{F}_{k+1} = \mathcal{F}_k \backslash \{\boldsymbol{\beta}\}$, $k = k + 1$, and go to Step 1.

(This algorithm may be very primitive, and more efficient algorithms need to be developed.)

Let $\mathbb{S}^\mathcal{F}$ denote the space of $|\mathcal{F}| \times |\mathcal{F}|$ symmetric matrices whose row and column indices are represented by $\mathcal{F}$. We use the notation $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F})$ to denote a matrix in $\mathbb{S}^\mathcal{F}$. We identify the $|\mathcal{H}|$-dimensional vector $(\boldsymbol{x}^{\boldsymbol{\gamma}} : \mathcal{H})$ with the rank-1 matrix $(\boldsymbol{x}^{\boldsymbol{\alpha}} : \mathcal{F})(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{F})^T \in \mathbb{S}^\mathcal{F}$, which is denoted by $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \square\mathcal{F})$. More precisely, each element $\boldsymbol{x}^{\boldsymbol{\gamma}}$ of the vector $(\boldsymbol{x}^{\boldsymbol{\gamma}} : \mathcal{H})$ is identified with the set of elements of the matrix $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \square\mathcal{F})$ placed at $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-th positions satisfying $\boldsymbol{\alpha} \in \mathcal{F}$, $\boldsymbol{\beta} \in \mathcal{F}$ and $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$. Therefore, if $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F}) = (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \square\mathcal{F}) \in \mathbb{S}^\mathcal{F}$, then

$$w_{\boldsymbol{\alpha}_1\boldsymbol{\beta}_1} \;=\; w_{\boldsymbol{\alpha}_2\boldsymbol{\beta}_2} \text{ if } \boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2. \tag{30}$$

This is an important property of the matrix $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \square\mathcal{F}) \in \mathbb{S}^\mathcal{F}$, and each matrix of the moment matrix cone defined in the following inherits this property.

We introduce the moment matrix cone, a symmetric matrix representation of the moment cone $\mathbb{M}(\mathcal{H},\mathbb{R}^n_+)$ defined by (12) with $\mathcal{H}=\mathcal{F}+\mathcal{F}$ and $\mathbb{L}=\mathbb{R}^n_+$:

$$
\begin{array}{l}
\mathbb{M}^d(\Box\mathcal{F},\mathbb{R}^n_+) \\
\quad = \left\{ \displaystyle\sum_{p=1}^{q} ((\boldsymbol{x}_p)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}} : \boldsymbol{x}_p \in \mathbb{R}^n_+ \ (p=1,\ldots,q) \text{ and } q \in \mathbb{Z}_+ \right\}.
\end{array}
$$

(The superscript d is used to mean "a dense moment matrix cone" in comparison to "a sparse moment matrix cone", which is introduced in the next section). The polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x},\mathcal{H}]$ $(j \in J_0)$ is rewritten as

$$
\begin{array}{rcl}
\psi(\boldsymbol{x}) & = & (\bar{\psi}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}), \\
h_j(\boldsymbol{x}) & = & ((\bar{h}_j)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}) \ (j \in J_0)
\end{array}
\tag{31}
$$

for some $(\bar{\psi}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}),\ ((\bar{h}_j)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}$ $(j \in J_0)$. Here $(v_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F})$ denotes the inner product $\sum_{\boldsymbol{\alpha}\in\mathcal{F}}\sum_{\boldsymbol{\beta}\in\mathcal{F}} v_{\boldsymbol{\alpha\beta}}\, w_{\boldsymbol{\alpha\beta}}$ of $(v_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}$ and $(w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}$. Then, we obtain a moment matrix cone problem

$$
\begin{array}{ll}
\text{minimize} & (\bar{\psi}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \\
\text{subject to} & ((\bar{h}_0)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) = 1, \\
& ((\bar{h}_j)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) = 0 \ (j \in J), \\
& (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{M}^d(\Box\mathcal{F},\mathbb{R}^n_+),
\end{array}
\tag{32}
$$

which is equivalent to (13) with $\mathcal{H}=\mathcal{F}+\mathcal{F}$, and attains the optimal objective of POP (1) under conditions (9), (14), (15) and (16) by Corollary 3.3.

Let $\mathbb{S}^{\mathcal{F}}_+$ denote the cone of positive semidefinite matrix in $\mathbb{S}^{\mathcal{F}}$, and $\mathbb{N}^{\mathcal{F}}$ the cone of matrices of nonnegative elements in $\mathbb{S}^{\mathcal{F}}$. Notice that every $(w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{M}^d(\Box\mathcal{F},\mathbb{R}^n_+)$ lies in $\mathbb{S}^{\mathcal{F}}_+ \bigcap \mathbb{N}^{\mathcal{F}}$ and satisfies (30). (Recall that $(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}) = (\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{F})(\boldsymbol{x}^{\boldsymbol{\beta}} : \mathcal{F})^T$ by definition). Thus, a doubly nonnegative cone relaxation of POP (1) is obtained as a relaxation of (32):

$$
\begin{array}{ll}
\text{minimize} & (\bar{\psi}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \\
\text{subject to} & ((\bar{h}_0)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) = 1, \\
& ((\bar{h}_j)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) = 0 \ (j \in J), \\
& \text{the condition (30)},\ (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}_+ \bigcap \mathbb{N}^{\mathcal{F}}.
\end{array}
\tag{33}
$$

If $(w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F})$ is a feasible solution of (32), then it is a feasible solution of (33). The converse is not true in general. Therefore, (33) serves as a relaxation for POP (1) for computing a lower bound of the minimum objective value, but the lower bound may not attain the exact minimum objective value in general.

For the example (7) given in Section 2.1, take

$$
\mathcal{F} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\},
\tag{34}
$$

so that $\mathcal{F}$ satisfies $\mathcal{H}_{\min} \subset \mathcal{F}+\mathcal{F}$ with $\tau = 4$. The variable matrix $(w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}$ becomes a 5×5 matrix. It is easy to check that $\mathcal{F}+\mathcal{F}$ consists of 14 elements. Thus, if elements of the variable matrix are identified by the condition (30), the matrix involves 14 independent real variables $w_{\boldsymbol{\alpha\beta}}$ ($\boldsymbol{\alpha} \in \mathcal{F},\ \boldsymbol{\beta} \in \mathcal{F},\ \boldsymbol{\alpha}+\boldsymbol{\beta} \in \mathcal{H}$).

Although (33) is called a dense doubly nonnegative cone relaxation for POP (1) in the next section, sparsity involved in (1) has already been considered. In other words, we can expect that the size of variable matrix $(w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}$ and the number of independent variables become smaller as the size of the union of support sets $\mathcal{H}_{\min} = \mathrm{supp}(\psi) \bigcup \left(\bigcup_{j \in J_0} \mathrm{supp}(h_j) \right)$ decreases.

6.2 Further exploitation of sparsity

The method in Section 6.1 can be easily extended to a more general framework: $\mathcal{H} = \bigcup_{s=1}^{t} (\mathcal{F}_s + \mathcal{F}_s)$ for some $\mathcal{F}_s \subset \mathbb{Z}^n_+$ $(s = 1, \ldots, t)$. We require condition

$$\mathcal{H}_{\min} \subset \bigcup_{s=1}^{t} (\mathcal{F}_s + \mathcal{F}_s) \tag{35}$$

We note that the family $\{\mathcal{F}_1, \ldots, \mathcal{F}_t\}$ are not necessary to be disjoint; some $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$ with $1 \leq s_1 < s_2 \leq t$ can have a nonempty intersection. Let $\mathcal{F} = \bigcup_{s=1}^{t} \mathcal{F}_s \subset \mathbb{Z}^n_+$. Then $\mathcal{F}$ satisfies $\mathcal{H}_{\min} \subset \mathcal{F}+\mathcal{F}$. We implicitly assume that the polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ $(j \in J_0)$ are sparse such that $\mathcal{H} = \bigcup_{s=1}^{t} (\mathcal{F}_s + \mathcal{F}_s)$ can be chosen with the size of much smaller than the sizes of $\mathcal{H}_{\max}$ and $\mathcal{F}+\mathcal{F}$. We present a method for generating $\mathcal{F}_s$ $(s = 1, \ldots, t)$ at the end of this section.

We now replace $\mathbb{M}^d(\Box\mathcal{F}, \mathbb{R}^n_+)$ by a sparse moment matrix cone

$$\begin{aligned} & \mathbb{M}^s(\Box\mathcal{F}_1, \ldots, \Box\mathcal{F}_t, \mathbb{R}^n_+) \\ = & \left\{ \sum_{p=1}^{q} \left(((\boldsymbol{x}_p)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}_1), \ldots, ((\boldsymbol{x}_p)^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}_t) \right) \right. \\ & \qquad \left. : \boldsymbol{x}_p \in \mathbb{R}^n_+ \ (p = 1, \ldots, q) \text{ and } q \in \mathbb{Z}+ \right\}, \end{aligned}$$

and the representation (31) of the polynomials $\psi,\ h_j \in \mathbb{R}[\boldsymbol{x}, \mathcal{H}]$ $(j \in J_0)$ by

$$\begin{aligned} \psi(\boldsymbol{x}) &= \sum_{s=1}^{t} ((\bar{\psi}_s)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}_s), \\ h_j(\boldsymbol{x}) &= \sum_{s=1}^{t} ((\bar{h}_{sj})_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \Box\mathcal{F}_s) \ (j \in J_0) \end{aligned}$$

for some $((\bar{\psi}_s)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s),\ ((\bar{h}_{sj})_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \in \mathbb{S}^{\mathcal{F}_s}$ $(s = 1, \ldots, t,\ j \in J_0)$. Then, a sparse moment matrix cone problem equivalent to POP (1) is obtained as

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{s=1}^{t}((\bar{\psi}_s)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \\ \text{subject to} & \displaystyle\sum_{s=1}^{t}((\bar{h}_{s0})_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) = 1, \\ & \displaystyle\sum_{s=1}^{t}((\bar{h}_{sj})_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) = 0 \ (j \in J), \\ & ((w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_1), \ldots, (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_t)) \in \mathbb{M}^s(\Box\mathcal{F}_1, \ldots, \Box\mathcal{F}_t, \mathbb{R}^n_+), \end{array} \tag{36}$$

and a sparse doubly nonnegative cone relaxation of POP (1) is given by

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{s=1}^{t}((\bar{\psi}_s)_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \\ \text{subject to} & \displaystyle\sum_{s=1}^{t}((\bar{h}_{s0})_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) = 1, \\ & \displaystyle\sum_{s=1}^{t}((\bar{h}_{sj})_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \bullet (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) = 0 \ (j \in J), \\ & \text{the condition (30)}, \ (w_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_s) \in \mathbb{S}^{\mathcal{F}_s}_+ \bigcap \mathbb{N}^{\mathcal{F}_s} \ (s = 1, \ldots, t). \end{array} \tag{37}$$

Since $\mathcal{F} = \sum_{s=1}^{t} \mathcal{F}_s$ satisfies $\mathcal{H}_{\min} \subset \mathcal{F} + \mathcal{F}$, the dense moment cone matrix relaxation (32) and doubly nonnegative cone relaxation (33) can be applied to POP (1). The moment matrix cone relaxations (32) and (36) are equivalent in the sense that both attain the exact optimal value of POP (1) under conditions (9), (14), (15) and (16). However, the sparse doubly nonnegative cone relaxation (37) may not be as effective as the dense one (33); the lower bound obtained by (37) for the optimal objective value of POP (1) may be inferior to the lower bound by (33). In fact, if $(\bar{w}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F})$ is a feasible solution of (33), we can construct a feasible solution $((\hat{w}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_1), \ldots, (\hat{w}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F}_t))$ with the same objective value as the feasible solution $(\bar{w}_{\boldsymbol{\alpha\beta}} : \Box\mathcal{F})$ of (33). The converse is not true in general. The advantage of (37) over (33) is its size, which makes more efficient to compute a lower bound of the optimal objective value of POP (1).

For the example (7) given in Section 2.1, let

$$\mathcal{F}_1 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ and } \mathcal{F}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\},$$

which are obtained by Algorithm 6.2 with $\mathcal{F}$ given by (34). Then

$$\mathcal{F}_1 + \mathcal{F}_1 = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$
\begin{aligned}
\mathcal{F}_2 + \mathcal{F}_2 &= \left\{ \begin{pmatrix} 2\\2\\0\\0 \end{pmatrix} \begin{pmatrix} 1\\1\\2\\0 \end{pmatrix} \begin{pmatrix} 1\\1\\0\\2 \end{pmatrix} \begin{pmatrix} 0\\0\\4\\0 \end{pmatrix} \begin{pmatrix} 0\\0\\2\\2 \end{pmatrix} \begin{pmatrix} 0\\0\\0\\4 \end{pmatrix} \right\}, \\
\bigcup_{s=1}^{2} \mathcal{F}_s &= \mathcal{F}, \ \bigcap_{s=1}^{2} \mathcal{F}_s = \emptyset, \\
\bigcup_{s=1}^{2} (\mathcal{F}_s + \mathcal{F}_s) &= \mathcal{H}_{\min}, \ \bigcap_{s=1}^{2} (\mathcal{F}_s + \mathcal{F}_s) = \left\{ \begin{pmatrix} 2\\2\\0\\0 \end{pmatrix} \right\}.
\end{aligned}
$$

The sparse doubly nonnegative cone relaxation problem (37) involves two variable matrices, 2×2 $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F}_1)$ and 3×3 $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F}_2)$. If the elements of the variable matrices are identified by the condition (30), the matrix involves $|\mathcal{H}_{\min}| = 8$ independent real variable, while the dense matrix cone relaxation problem (33) involves 14 independent variables, as seen in Section 6.1.

We conclude this section by presenting a method for computing $\mathcal{F}_s \subset \mathbb{Z}^n$ $(s = 1, \ldots, t)$ that satisfies the condition (35).

Algorithm 6.2

Step 0: Choose a $\mathcal{F} \subset \mathbb{Z}^n_+$ that satisfies $\mathcal{H}_{\min} \subset \mathcal{F} + \mathcal{F}$. (Algorithm 6.1 can be applied in advance).

Step 1: Construct an undirected graph $G(\mathcal{F}, \mathcal{E})$ with the node set $\mathcal{F}$ and the edge set $\mathcal{E} \subset \mathcal{F} \times \mathcal{F}$ given by $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{E}$ if and only if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathcal{H}_{\min}$. $((\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{E}$ is identified with $(\boldsymbol{\beta}, \boldsymbol{\alpha}) \in \mathcal{E}$ since the graph $G(\mathcal{F}, \mathcal{E})$ is undirected).

Step 2: Choose the maximal cliques $\mathcal{F}_s$ $(s = 1, \ldots, t)$.

In general, finding all maximal cliques of a given graph is a difficult problem. To avoid this difficulty, the following step may be inserted between Step 1 and 2.

Step 1.5: Replace the graph $G(\mathcal{F}, \mathcal{E})$ by a chordal extension of $G(\mathcal{F}, \mathcal{E})$.

Finding the maximal cliques of a chordal graph is a tractable problem and the number of its maximal cliques is bounded by $|\mathcal{F}|$. In addition, this step strengthens the effectiveness of the doubly nonnegative cone relaxation of POP (1). Step 1.5 was originally proposed for the sparse version [21] of Lasserre's SDP relaxation [14].

6.3 Incorporation of the doubly nonnegative cone relaxation into Lasserre's SDP relaxation

If the doubly nonnegative condition $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}_+ \bigcap \mathbb{N}^{\mathcal{F}}$ is replaced by an SDP condition $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{F}) \in \mathbb{S}^{\mathcal{F}}_+$ in (33), the resulting problem may be regarded as a variant of Lasserre's SDP relaxation [14] (with the lowest hierarchy) applied to POP (1). In the previous discussions, we have focused on the POP of the form (1) satisfying conditions (9), (14), (15) and (16) to theoretically ensure that the resulting moment cone relaxations (13), (32) and (36) are exact, and derived the doubly nonnegative cone relaxations (33) and (37) for POP (1). But, the doubly nonnegative cone relaxation can be directly applied to a fairly general POP with nonnegative variables, and incorporated into Lasserre's SDP relaxation [14].

Let $J = \{1, \ldots, \ell\}$, $J_0 = \{0\} \bigcup J$, ψ, $g_j \in \mathbb{R}[\boldsymbol{x}]$ $(j \in J_0)$ and $\boldsymbol{x} \in \mathbb{R}^n$. Consider a POP

$$\text{minimize} \quad g_0(\boldsymbol{x}) \quad \text{subject to} \quad g_j(\boldsymbol{x}) \geq 0 \ (j \in J). \tag{38}$$

An equality constraint $g(\boldsymbol{x}) = 0$ with $g \in \mathbb{R}[\boldsymbol{x}]$ can be included in POP (38) as two inequality constraints $g(\boldsymbol{x}) \geq 0$ and $-g(\boldsymbol{x}) \geq 0$. Let

$$\begin{array}{rcl} \omega_j &=& \lceil \deg(g_j)/2 \rceil \ (j \in J_0), \ \omega_{\max} = \max\{\omega_j \ (j \in J_0)\}, \\ \mathcal{A}_\eta &=& \left\{\boldsymbol{\alpha} \in \mathbb{Z}^n_+ : |\boldsymbol{\alpha}|_1 \leq \eta\right\} \ \text{for every } \eta \in \mathbb{Z}_+. \end{array}$$

Choose $\omega \in \mathbb{Z}_+$ not less than $\omega_{\max}$. Then, POP (38) is equivalent to the polynomial SDP

$$\begin{array}{ll} \text{minimize} & g_0(\boldsymbol{x}) \\ \text{subject to} & g_j(\boldsymbol{x})(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \mathcal{A}_{\omega-\omega_j})(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \mathcal{A}_{\omega-\omega_j})^T \in \mathbb{S}^{\mathcal{A}_{\omega-\omega_j}}_+ \ (j \in J), \\ & (\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \mathcal{A}_\omega)(\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} : \mathcal{A}_\omega)^T \in \mathbb{S}^{\mathcal{A}_\omega}_+. \end{array} \tag{39}$$

The problem (39) can be rewritten as

$$\begin{array}{ll} \text{minimize} & ((c_0)_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \bullet (\boldsymbol{x}^{\alpha+\beta} : \square\mathcal{A}_\omega) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\gamma}\in\mathcal{A}_\omega} \sum_{\boldsymbol{\delta}\in\mathcal{A}_\omega} \left((c_{j\boldsymbol{\gamma\delta}})_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_{\omega-\omega_j}\right) \boldsymbol{x}^{\boldsymbol{\gamma}+\boldsymbol{\delta}} \in \mathbb{S}^{\mathcal{A}_{\omega-\omega_j}}_+ \ (j \in J), \\ & (\boldsymbol{x}^{\alpha+\beta} : \square\mathcal{A}_\omega) \in \mathbb{S}^{\mathcal{A}_\omega}_+, \end{array} \tag{40}$$

for some $((c_0)_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \in \mathbb{S}^{\mathcal{A}_\omega}$, $\left((c_{j\boldsymbol{\gamma\delta}})_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_{\omega-\omega_j}\right) \in \mathbb{S}^{\mathcal{A}_{\omega-\omega_j}}$ $(j \in J, \ \boldsymbol{\gamma} \in \mathcal{A}_\omega, \ \boldsymbol{\delta} \in \mathcal{A}_\omega)$.

Each $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) = (\boldsymbol{x}^{\alpha+\beta} : \square\mathcal{A}_\omega)$ satisfies not only $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \in \mathbb{S}^{\mathcal{A}_\omega}_+$, but also (30). Thus, we have Lasserre's SDP relaxation of POP (38) with the hierarchy level ω:

$$\begin{array}{ll} \text{minimize} & ((c_0)_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \bullet (w_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\gamma}\in\mathcal{A}_\omega} \sum_{\boldsymbol{\delta}\in\mathcal{A}_\omega} \left((c_{j\boldsymbol{\gamma\delta}})_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_{\omega-\omega_j}\right) w_{\boldsymbol{\gamma\delta}} \in \mathbb{S}^{\mathcal{A}_{\omega-\omega_j}}_+ \ (j \in J), \\ & \text{the condition (30)}, \ (w_{\alpha\beta} : \square\mathcal{A}_\omega) \in \mathbb{S}^{\mathcal{A}_\omega}_+. \end{array} \tag{41}$$

See [14] for more details. It is shown in [14] that the optimal value of SDP (41) converges to the optimal value of POP (38) monotonically as $\omega \to \infty$ under a certain moderate assumption that requires the boundedness of the feasible region of POP (38).

Now suppose that $x_i \geq 0$ $(i = 1, \ldots, n)$ are included in the inequality constraints $g_j(\boldsymbol{x}) \geq 0$ $(j \in J)$ of POP (38). In this case, $\boldsymbol{x}^{\boldsymbol{\gamma}} \geq 0$ for every feasible solution $\boldsymbol{x}$ of (38) and $\boldsymbol{\gamma} \in \mathbb{Z}^n_+$. Therefore, the positive semidefinite matrix cones $\mathbb{S}^{\mathcal{A}_{\omega-\omega_j}}_+$ $(j \in J)$ and

$\mathbb{S}_+^{\mathcal{A}_\omega}$ can be replaced by their intersections with $\mathbb{N}^{\mathcal{A}_{\omega-\omega_j}}$ $(j \in J)$ and $\mathbb{N}^{\mathcal{A}_\omega}$, respectively, in (39), (40) and (41). The resulting doubly nonnegative cone relaxation of (38) is:

$$\begin{array}{ll} \text{minimize} & ((c_0)_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \bullet (w_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega) \\ \text{subject to} & \displaystyle\sum_{\boldsymbol{\gamma}\in\mathcal{A}_\omega} \sum_{\boldsymbol{\delta}\in\mathcal{A}_\omega} \left((c_{j\boldsymbol{\gamma\delta}})_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_{\omega-\omega_j}\right) w_{\boldsymbol{\gamma\delta}} \in \mathbb{S}_+^{\mathcal{A}_{\omega-\omega_j}} \bigcap \mathbb{N}^{\mathcal{A}_{\omega-\omega_j}} \ (j \in J), \\ & \text{the condition (30)}, \ (w_{\alpha\beta} : \square\mathcal{A}_\omega) \in \mathbb{S}_+^{\mathcal{A}_\omega} \bigcap \mathbb{N}^{\mathcal{A}_\omega}. \end{array} \tag{42}$$

The doubly nonnegative cone relaxation (42) is at least as strong as the SDP relaxation (41) for POP (38), but its computational costs are higher than (41).

For the example (6) given in Section 2.1, we have $\omega_{\max} = 2$. If the lowest hierarchy level $\omega = \omega_{\max} = 2$ is chosen, then $\mathcal{A}_\omega = \left\{\boldsymbol{\alpha} \in \mathbb{Z}^3 : |\boldsymbol{\alpha}|_1 \le \omega\right\}$ consists of 10 elements, and the variable matrix $(w_{\boldsymbol{\alpha\beta}} : \square\mathcal{A}_\omega)$ of the doubly nonnegative cone relaxation (42) is 10×10. Therefore, (42) is a much larger doubly nonnegative cone problem than (33) in Section 6.1 and (37) in Section 6.2, although it provides a slightly better lower bound for the optimal objective value of POP (1) than (33) and (37), as will be seen in the next section.

A sparse version of Lasserre's SDP relaxation [14] was proposed in [21] by Waki, Kim, Kojima and Muramatsu. See also [12]. The discussions in this section can be modified and extended so that doubly nonnegative cone relaxation can be incorporated into the sparse version.

7 Concluding remarks

We have extended the results on the CPP relaxation for QOPs [1] to POP (1) that satisfies conditions (9), (14), (15) and (16) by introducing the moment cone (12) and the moment cone relaxation (13) of the POP. The moment cone relaxation provides the exact optimal value of the POP, but its numerical implementation is quite difficult. As a further relaxation that can be numerically tractable, the doubly nonnegative cone relaxation has been derived, and sparsity exploitation has been discussed.

It is interesting to see how the doubly nonnegative cone relaxation derived from the moment cone relaxation works in comparison to Lasserre's SDP relaxation, and to see how exploiting sparsity enhances the performance of the doubly nonnegative cone relaxation. We applied the following relaxations to the numerical example (6).

(a) Lasserre's SDP relaxation provided by SparsePOP [22].

(b) Doubly nonnegative cone relaxation (33).

(c) SDP relaxation obtained by replacing $\mathbb{S}_+^{\mathcal{F}} \bigcap \mathbb{N}^{\mathcal{F}}$ by $\mathbb{S}_+^{\mathcal{F}}$ in (33).

(d) Doubly nonnegative cone relaxation (37).

(e) SDP relaxation obtained by replacing $\mathbb{S}_+^{\mathcal{F}_s} \bigcap \mathbb{N}^{\mathcal{F}_s}$ by $\mathbb{S}_+^{\mathcal{F}_s}$ $(s = 1, \dots, t)$ in (37).

Relaxation SDP	LBD	Size of $\boldsymbol{A}$	Relaxation DN cone	LBD	Size of $\boldsymbol{A}$
(a)	-4.3050087e-1	(34,165)			
(c)	-4.0000000e+1	(3,25)	(b)	-4.3057829e-1	(13,35)
(e)	-4.0000000e+1	(3,13)	(d)	-4.3058400e-1	(7,17)

Table 1: Comparison of the five relaxations applied to POP (6).

The results are shown in Table 1.

Each relaxation problem was solved by SeDuMi [20] after it was converted to an SDP. "LBD" denotes the lower bound computed as the optimal value of the SDP for the unknown optimal value of the POP (6), and "Size of $\boldsymbol{A}$" indicates the size of the coefficient matrix $\boldsymbol{A}$ in the standard SDP format of SeDuMi. POP (6) is very small, so all SDPs were solved less than 0.2 seconds. A sparse version [21] of Lasserre's relaxation (a) was also applied to POP (6) but the results were the same as those of (a) because POP (6) did not satisfy the structured sparsity that can be exploited by the sparse version.

SparsePOP also provided an approximate optimal value -4.3050087e-1 to the unknown optimal value. Lasserre's relaxation (a) attained this bound (in 8 digits), a very accurate result. The two SDP relaxations (c) and (e) respectively derived from the moment cone relaxations (33) and (37) did not work effectively. We observe that the doubly nonnegative cone relaxations (b) and (d) are slightly less effective than (a), and that their Sizes of $\boldsymbol{A}$ are much smaller than that of (a). To evaluate the effectiveness and efficiency of the doubly nonnegative relaxations (b) and (d) in comparison to (a) and its sparse version [21], extensive numerical experiments are necessary. This will be a subject of our study in the future.

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