2.1 Informal development of the Fourier transform:

Definition 1. The space $L^1(\mathbb{R})$ is the set of all absolutely integrable functions defined in $\mathbb{R}$, that is,

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \|f\|_1 := \int_{\mathbb{R}} |f(x)|dx < \infty \right\}.$$ 

Let $f$ be a function in $L^1(\mathbb{R})$. We define the Fourier transform of $f$ by

$$\mathcal{F}f(\lambda) = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx.$$ 

The inverse Fourier transform of a function $g$ in $L^1(\mathbb{R})$ is defined by

$$\mathcal{F}^{-1}g(x) = \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\lambda)e^{-ix\lambda}d\lambda.$$ 

Note: Let $f \in L^1(\mathbb{R})$. For any $\lambda \in \mathbb{R}$, $|\hat{f}(\lambda)| \leq \|f\|_1$, that is, $\|f\|_\infty \leq \|f\|_1$ where $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$.

- See Examples 2.2, 2.5.

2.2 Properties of the Fourier transform:

Theorem 1. Let $f \in L^1(\mathbb{R})$ and $f$ be a continuously differentiable function. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{ix\lambda}d\lambda$$

with

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\lambda}dx.$$ 

That is, $\mathcal{F}^{-1}\mathcal{F}f = f$.

Theorem 2. Let $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$. Then, we have the following properties:

1. $\mathcal{F}(af + bg) = a\mathcal{F}f + b\mathcal{F}g$ and $\mathcal{F}^{-1}(af + bg) = a\mathcal{F}^{-1}f + b\mathcal{F}^{-1}g$.
2. Assume that $x^n f(x) \in L^1(\mathbb{R})$. Then,

$$[(\cdot)^n f](\lambda) = [(iD)^n \hat{f}](\lambda).$$

3. Assume that $f^{(n)}(x) \to 0$ as $x \to \pm \infty$ and $f^{(n)} \in L^1(\mathbb{R})$. Then,

$$\hat{f}^{(n)}(\lambda) = (i\lambda)^n \hat{f}(\lambda).$$

4. $\hat{f}(\cdot - a)(\lambda) = e^{-i\lambda a} \hat{f}(\lambda).$
5. \( \hat{f}(b\cdot)(\lambda) = \hat{f}(\lambda/b)/b \) where \( b \neq 0 \).

**Definition 2.** Suppose that \( f, g \in L^2(\mathbb{R}) \). The convolution of \( f \) and \( g \) is defined by
\[
(f \ast g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt = \int_{\mathbb{R}} f(t)g(x-t)dt.
\]

**Theorem 3.** Let \( f, g \in L^2(\mathbb{R}) \). Then
\[
\hat{f} \ast \hat{g} = \sqrt{2\pi} \hat{f} \hat{g}.
\]
Also, \( \mathcal{F}^{-1}(\hat{f} \hat{g}) = \frac{1}{\sqrt{2\pi}} f \ast g \).

**Theorem 4.** (Plancherel) Let \( f, g \in L^2(\mathbb{R}) \). Then
\[
\int_{\mathbb{R}} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}d\lambda = \int_{\mathbb{R}} f(x)\overline{g(x)}dx.
\]
In particular, \( \|\hat{f}\|_2 = \|f\|_2 \).

### 2.3 The Sampling Theorem

**Definition 3.** A function \( f \) is said to be band-limited if there exists \( L > 0 \) such that \( \hat{f}(\lambda) = 0 \) for any \( |\lambda| > L \). The frequency \( \nu := \frac{L}{2\pi} \) is called the Nyquist frequency and \( 2\nu = \frac{L}{\pi} \) is the Nyquist rate.

**Theorem 5.** (Shannon-Whittaker Sampling Theorem) Suppose that \( \hat{f} \) is piecewise smooth, continuous function and \( \hat{f}(\lambda) = 0 \) for \( |\lambda| > L \) with \( L > 0 \). Then,
\[
f(x) = \sum_{j \in \mathbb{Z}} f\left(\frac{j\pi}{L}\right) \frac{\sin(Lx - j\pi)}{Lx - j\pi}.
\]

**Example:** Let \( a > 0 \). If \( G(x) = e^{-ax^2} \), then
\[
\hat{G}(\lambda) = \frac{1}{\sqrt{2a}} e^{-\lambda^2/4a}.
\]
If \( a = 1/2 \), \( G(x) = e^{-x^2/2} \) and \( \hat{G}(\lambda) = e^{-\lambda^2/2} \).