Chapter 1 The Real Numbers: Sets, Sequences, and Functions

□ Sets

Definition. A set $E$ in $\mathbb{R}$ is dense in $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ ($a < b$), there is $c \in E$ such that $a < c < b$.

Theorem 1. The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Equipotent sets: We call two sets $A$ and $B$ equipotent if there is a one-to-one mapping $f$ from $A$ and $B$, denoted by $A \cong B$.

Definition Let $\mathbb{N}_n = \{1, 2, \ldots, n\} \subset \mathbb{N}$. For any set $A$, we say:

1. A set $E$ is finite if $E$ is empty, i.e., $E = \phi$ or it has only a finite number of elements in $E$. That is, there is a number $n$ for which $E$ is equipotent to $\mathbb{N}_n$.
2. $E$ is infinite if $E$ is NOT finite.
3. $E$ is countably infinite if $E \cong \mathbb{N}$. Countable sets are sometimes called enumerable or denumerable sets.
4. $E$ is (at most) countable if $E$ is finite or countable.
5. $E$ is uncountable if $E$ is neither finite nor countable.

Theorem 2. The following sets are countable.

1. $\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$
2. The set of rational numbers $\mathbb{Q}$.
3. The union of countable collection of countable sets. That is, if $\{E_n\}$ be a sequence of countable sets, then $E = \bigcup_{n=1}^{\infty} \{E_n\}$ is countable.

Theorem 3. The sets $(a, b)$, $[a, b)$ and $[a, b]$ are uncountable.

□ Sequence of Real Numbers

Sequence: A sequence of real numbers is a function on $\mathbb{N}$ to $\mathbb{R}$

$$a : \mathbb{N} \ni n \mapsto a(n) =: a_n.$$ 

Limit point: Let $(a_n)$ be a sequence of real numbers. Then, $a$ is called a limit of $(a_n)$ i.e.,

$$\lim_{n \to \infty} a_n = a, \text{ if for any } \epsilon > 0, \text{ there exists an integer } N > 0 \text{ such that }$$

$$|a_n - a| < \epsilon, \quad \forall n \geq N.$$ 

Let $a$ be the limit of $(a_n)$. Then, for any $\epsilon > 0$, all but a finite number of $a_n$ are in the ball $B(x, \epsilon)$.

Theorem 4. (Bolzano-Weierstrass Theorem) Every bounded sequence of real numbers has a convergent subsequence.

Theorem 5. A subset $E$ of $\mathbb{R}$ is closed if and only if it contains all its limit points.

Theorem 6. A monotone sequence converges if and only if it is bounded.

Cauchy Sequence: A sequence $(a_n)$ is called Cauchy sequence if for any $\epsilon > 0$, there exists an integer $N > 0$ such that

$$|a_n - a_m| < \epsilon, \quad \forall n, m \geq N.$$
Theorem 7. A sequence of real numbers converges if and only if it is a Cauchy sequence.

Cluster Point: Let \((a_n)\) be a sequence of real numbers. Then, \(a\) is called a cluster point of \((a_n)\) if for any \(\epsilon > 0\) and \(N > 0\), there exists an integer \(n \geq N\) such that
\[|a_n - a| < \epsilon.\]

For instance, consider \(a_n = (-1)^n\).

\(\text{limsup/inf}\): Let \((a_n)\) be a sequence of real numbers. Define
\[\limsup_{n} a_n = \inf_{n} \sup_{k \geq n} a_k = \limsup_{n} a_n.\]

Then, \(\ell = \limsup a_n\) iff (i) For any \(\epsilon > 0\), there exists \(n \in \mathbb{N}\) such that \(a_k < \ell + \epsilon\) for all \(k \geq n\) and (ii) given \(\epsilon > 0\), and \(n\), there exists \(k \geq n\) such that \(a_k > \ell - \epsilon\).

Similarly, we can define
\[\liminf_{n} a_n = \sup_{n} \inf_{k \geq n} a_k = \liminf_{n} a_n.\]

Properties: Let \((a_n)\) be a sequence of real numbers. Then
\[
\begin{align*}
(1) \quad & \lim (-a_n) = -\lim a_n. \\
(2) \quad & \lim a_n \leq \liminf a_n. \\
(3) \quad & \text{If } a_n \to \ell, \text{ then } \ell = \liminf a_n = \limsup a_n. \\
\end{align*}
\]

\(\square\) Open and Closed Sets of Real Numbers

Properties:
\[
\begin{align*}
(1) \quad & \text{The intersection of any finite collection of open sets is open.} \\
(2) \quad & \text{Every open set of real numbers is the union of a countable collection of disjoint open intervals.} \\
(3) \quad & \text{The union of any collection of open sets is open} \\
(4) \quad & \text{The union of any finite collection of closed sets is closed.} \\
(5) \quad & \text{The intersection of any collection of closed sets is closed.} \\
\end{align*}
\]

Theorem 8. (Heine-Borel) Let \(F\) be a closed and bounded set in \(\mathbb{R}\). Then, each open covering of \(F\) has a finite subcovering.

Continuity: A real-valued function \(f\) defined on \(E\) is said to continuous at \(x \in E\) if given \(\epsilon > 0\), there exists \(\delta > 0\) such that for all \(y\) with \(|x - y| < \delta\), we have
\[|f(x) - f(y)| < \epsilon.\]

Uniform Continuity: A real-valued function \(f\) defined on \(E\) is said to uniformly continuous (on \(E\)) if given \(\epsilon > 0\), \(\exists \delta > 0\) such that for all \(x, y\) with \(|x - y| < \delta\), we have
\[|f(x) - f(y)| < \epsilon.\]

A continuous function on a closed and bounded set is uniformly continuous.
**Lipschitz Continuity**: A real-valued function $f$ defined on $E$ is said to be **Lipschitz** (on $E$) if $\exists c > 0$ such that for all $x, y$ in $E$.

$$|f(x) - f(y)| \leq c|x - y|$$

**Theorem 9. (Extreme Value Theorem)** A continuous function on a closed and bounded set takes a minimum and a maximum values.

**Theorem 10. (Intermediate Value Theorem)** Let $f$ be a continuous function on $[a, b]$ and $f(a) < c < f(b)$. Then, there is number $x_0 \in (a, b)$ such that $f(x_0) = c$. set takes a minimum and a maximum values.

**Pointwise Convergence**: Let $(f_n)$ be a sequence of function defined on $E$. Then $(f_n)$ is said to converge **pointwise** if for every $x \in E$, we have $f(x) = \lim f_n(x)$ : that is, if given $x \in E$ and $\epsilon > 0$, $\exists N$ such that

$$|f(x) - f_n(x)| < \epsilon \quad \forall n \geq N.$$ 

**Uniform Convergence**: A sequence $(f_n)$ of function defined on $E$ is said to converge **uniformly** if given $\epsilon > 0$, $\exists N$ such that

$$|f(x) - f_n(x)| < \epsilon \quad \forall x \in E, \quad n \geq N.$$