Chapter 6 Differentiation and Integration

6.2 Differentiation of Monotone Functions

Definition 1. Let \( \mathcal{F} \) be a collection of closed and bounded intervals. We say that \( \mathcal{F} \) covers a set \( E \) in the sense of Vitali if \( \forall \epsilon > 0 \) and \( \forall x \in E \), there exist \( I \in \mathcal{J} \) such that \( x \in I \) and \( \ell(I) < \epsilon \).

Lemma 1. (Vitali) Let \( m(E) < \infty \) and \( \mathcal{F} \) be a collection of intervals that covers \( E \) in the sense of Vitali. Then, for any \( \epsilon > 0 \), there exist \( I_1, \ldots, I_N \) in \( \mathcal{J} \) such that

\[
m^*(E \sim \bigcup_{n=1}^{N} I_n) < \epsilon.
\]

Define the upper and lower derivatives of \( f \) at \( x \) respectively as:

\[
\overline{D}f(x) = \lim_{h \to 0^+} \sup_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{h}
\]

\[
\underline{D}f(x) = \lim_{h \to 0^+} \inf_{0 < |t| \leq h} \frac{f(x+t) - f(x)}{h}
\]

Clearly \( \overline{D}f(x) \geq \underline{D}f(x) \). If \( \overline{D}f(x) = \underline{D}f(x) \), we say that \( f \) is differentiable at \( x \).

Theorem 1. Lebesgue’s Theorem) If \( f \) is a monotone real-valued function on \([a, b]\), then \( f \) is differentiable a.e. on \((a, b)\).

Theorem 2. Let \( f \) be increasing real-valued function on \([a, b]\). Then,

1. \( f \) is differentiable a.e.
2. \( f' \) is measurable and \( \int_{a}^{b} f'(x)dx \leq f(b) - f(a) \).

6.3 Functions of Bounded Variation: Jordan’s Theorem

Let \( f \) be a real-valued function defined on \([a, b]\) and let \( P = \{x_0, \ldots, x_k\} \) be a partition of \([a, b]\), i.e., \( a = x_0 < x_1 < \cdots < x_k = b \). Define the variation of \( f \) with respect to \( P \) by

\[
V(f, P) = \sum_{n=1}^{k} |f(x_i) - f(x_{i-1})|
\]

and the total variation of \( f \) on \([a, b]\) by taking suprema over all subdivision of \([a, b]\) as

\[
TV(f) = \sup\{V(f, P) | P \text{ a partition of } [a, b]\}
\]

For a subinterval \([c, d] \subset [a, b]\), \( TV(f|_{[c,d]}) \) denotes the total variation of \( f|_{[c,d]} \).

Definition 2. A function \( f \) is called of Bounded Variation over \([a, b]\), if \( TV(f) < \infty \), which is denoted it by

\[
f \in BV[a, b].\]
Example Let \( f \) be a Lipschitz function on \([a, b]\), i.e.,
\[
|f(u) - f(v)| \leq C|u - v|, \quad \forall u, v \in [a, b].
\]
Then \( TV(f) \leq C|b - a| \).

Example Let \( f \) be a real-valued function defined on \([0, 1]\) as follows:
\[
f(x) = \begin{cases} 
x \cos(\pi/2x) & \text{if } 0 < x \leq 1, \\
0 & \text{if } x = 0.
\end{cases}
\]
Then \( f \) is continuous on \([0, 1]\), but NOT of bounded variation on \([0, 1]\).

Definition 3. The function \([a, b] \mapsto TV(f_{[a,x]}))\), where \(x \in [a, b]\), is called the total variation function for \( f \). Clearly, \( TV(f_{[a,x]}) \) is increasing on \([a, b]\).

Theorem 3. (Jordan’s Theorem) Let \( f \in BV\) on \([a, b]\) if and only if there exist two increasing functions \( g \) and \( h \) on \([a, b]\) such that \( f = g - h \).

Corollary 1. Let \( f \in BV[a,b] \). Then, \( f \) is differentiable a.e. in \((a,b)\) and \( f' \) is integrable over \([a,b]\). (see Theorem 2)

Proposition 1. Let \( f \) be an integrable function defined on \( E \). Then, for a given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for \( A \subset E \) with \( m(A) < \delta \),
\[
\int_A |f| < \epsilon.
\]

Let \( f \) be integrable on \([a, b]\) and define
\[
F(x) = \int_a^x f(t)dt.
\]

Lemma 2. Let \( F \) is continuous and \( F \in BV[a,b] \).

Lemma 3. Let \( f \) be an integrable function defined on \([a, b]\) and assume that \( \int_a^x f(t)dt = 0 \) for all \( x \in [a, b] \). Then \( f(x) = 0 \) a.e. in \([a, b]\).

Lemma 4. Let \( f \) be a bounded measurable function on \([a, b]\) and \( F(x) = \int_a^x f(t)dt + F(a) \). Then \( F'(x) = f(x) \) a.e. \( x \in [a, b] \).

Theorem 4. Let \( f \) be an integrable on \([a, b]\). Suppose that \( F(x) = \int_a^x f(t)dt + F(a) \). Then \( F'(x) = f(x) \) a.e. \( x \in [a, b] \).

6.4 Absolute Continuity

A real-valued function \( f \) is said to be absolutely continuous on \([a, b]\) if for a given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\sum_{i=1}^n |f(b_i) - f(a_i)| < \epsilon
\]
for every finite collection \( \{(a_i, b_i), \, i = 1, \ldots, n\} \) of nonoverlapping intervals with
\[
\sum_{i=1}^{n} |b_i - a_i| < \delta.
\]

**Remark.** If a function \( f \) is Lipschitz continuous on \([a, b]\), then \( f \) is absolutely continuous. But, there are absolutely continuous functions which are not Lipschitz continuous.

**Lemma 5.** If \( f \) is absolutely continuous on \([a, b]\), then \( f \) is bounded variation. Hence, \( f \) has derivative a.e.

**Theorem 5.** A function \( F \) is an indefinite integral, i.e., \( F(x) = \int_a^x F'(t) \, dt + F(a) \) if and only if \( F \) is absolutely continuous.