Chapter 3 Structures of Point Sets

3.1 Open and Closed Sets

**Definition** A set $O \subseteq \mathbb{R}$ is called **open** if for any $p \in O$,
\[ \exists \epsilon > 0 \text{ s.t. } N_{\epsilon}(p) \subseteq O. \]

A set $F \subseteq \mathbb{R}$ is called **closed** if $F^c$ is open.

**Remark:**
1. For any $x \in O$, $\exists \epsilon_x > 0$ such that $N_{\epsilon_x}(x) \subseteq O$, i.e.,
   \[ O = \bigcup_{x \in O} N_{\epsilon_x}(x). \]
   That is, any open set ($\neq \emptyset$) is represented as a (countable) union of open intervals.

2. For any collection $\{O_\alpha : \alpha \in A\}$ of open sets $O_\alpha$, $\cup_{\alpha \in A} O_\alpha$ is open.

3. For any collection $\{F_\alpha : \alpha \in A\}$ of closed sets $F_\alpha$, $\cap_{\alpha \in A} F_\alpha$ is closed.

4. For any finite collection $\{O_1, \ldots, O_n\}$ of open sets, $\cap_{j=1}^n O_j$ is open.

5. For any finite collection $\{F_1, \ldots, F_n\}$ of closed sets, $\cup_{j=1}^n F_j$ is closed.

**Recall:** Let $E$ be a subset of $\mathbb{R}$. A point $p \in \mathbb{R}$ is called a **limit point** of $E$ if for any $\epsilon > 0$, the neighborhood $N_{\epsilon}(p)$ contains a point $q \in E$ with $q \neq p$. That is, $p \neq q$ and $|p - q| < \epsilon$.

**Theorem 1.** A set $F$ is closed if and only if $F$ contains all its limit points.

3.2 Compact Sets

**Definition** Let $E$ be a subset of $\mathbb{R}$. A collection of open sets $\{O_\alpha\}_{\alpha \in A}$ is called an **open covering** of the set $E$ if
\[ E \subseteq \bigcup_{\alpha \in A} O_\alpha. \]

That is, for any $p \in E$, there exists $\alpha \in A$ such that $p \in O_\alpha$.

**Definition** A set $K \subseteq \mathbb{R}$ is called **compact** if every open covering $G = \{O_i\}$ of $K$ has a finite subcovering, that is,
\[ \exists O_{a_j} (j = 1, \ldots, n) \text{ s.t. } K \subseteq \bigcup_{j=1}^n O_{a_k}. \]

**Examples:**
1. Every finite set is compact.
2. The open interval $(0, 1)$ is NOT compact.
3. The set $[0, \infty)$ is NOT compact.

**Theorem 2.**
1. Every compact set in $\mathbb{R}$ is closed and bounded.
2. Every closed subset of a compact set is compact.
Theorem 3. (Heine-Borel Theorem). Every closed and bounded interval \([a, b]\) is compact.

Theorem 4. If \(S\) is an infinite subset of a compact set \(K\), then \(S\) has a limit point in \(K\).

Theorem 5. Let \(K\) be a compact set in \(\mathbb{R}\). Then the followings are equivalent.

1. \(K\) is closed and bounded.
2. \(K\) is compact.
3. Every infinite subset of \(K\) has a limit point in \(K\).