Chapter 6 The Riemann and Riemann-Stieltjes Integral

6.1 The Riemann Integral

□ **Review:**

- Let S be a subset of $\mathbb{R}$. Suppose that there exists $M \in \mathbb{R}$ such that (i) M is a upper bound of S; (ii) if u is an upper bound of S, then $M \leq u$. Then M is called the **least upper bounded** of S or the **supremum** of S. We write
$$M = \sup S.$$
- Suppose that there exists $m \in \mathbb{R}$ such that (i) m is a lower bound of S; (ii) if ℓ is a lower bound of S, then $\ell \leq m$. Then m is called the **greatest lower bounded** of S or the **infimum** of S. We write
$$m = \inf S.$$

□ **Notation :**

$$\begin{aligned} \mathcal{B} &= \{f : [a,b] :\to \mathbb{R} : \ f \text{ is bounded on } [a,b]\} \\ M &= \sup_{x \in [a,b]} f(x) = \sup\{f(x) : x \in [a,b]\} \\ m &= \inf_{x \in [a,b]} f(x) = \inf\{f(x) : x \in [a,b]\} \end{aligned}$$

Definition: A **partition** of $[a,b]$ is a set of points $P := \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \ldots, < x_{i-1} < x_i < \cdots < x_n = b. \tag{1}$$

For a given partition P, we use the following notation:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

for $i = 1, \ldots, n$. Clearly, $m \leq m_i \leq M_i \leq M$

Definition: For a given partition P on $[a,b]$ as in (1), the upper sum of f for P is to be

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

Also, the lower sum of f for P is defined to be

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

It is clear that $L(P,f) \leq U(P,f)$.

Definition: Let f be a bounded real-valued function on $[a,b]$. The upper integral of f for on $[a,b]$ is defined by

$$\overline{\int_a^b} f(x)dx = \inf_P U(P,f)$$

Also, the lower integral of f for on $[a,b]$ is defined to be

$$\underline{\int_a^b} f(x)dx = \sup_P L(P,f)$$

Lemma 1. *For any partition P on $[a,b]$,*

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

Definition A a bounded real-valued function f is said to be **Riemann integrable** on $[a, b]$ if

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

If f is Riemann integrable on $[a, b]$, we define the integral of f by

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

□ **Notation :**

$$\mathcal{R} = \{f : [a, b] :\to \mathbb{R} : \ f \text{ is Riemann integrable on } [a, b]\}$$

Definition:

(1) A partition P^* of $[a, b]$ is called a refinement of the partition P if each point of subdivision x_i of P is also a point of subdivision of P^*.
(2) The partition P^* is called a common refinement of P_1 and P_2 if P^* is a refinement of both P_1 and P_2.

Remark:

(1) Each pair of partitions P_1 and P_2 has common refinement (i.e., $P^* = P_1 \cup P_2$).
(2) Let $\tilde{P}$ be a common refinement of P_1 and P_2. Then $\tilde{P}$ is also a refinement of $P^* = P_1 \cup P_2$. So, P^* is (sometimes) called the first refinement of $P_1 \cup P_2$.
(3) The partition P^* is called a common refinement of P_1 and P_2 if P^* is a refinement of both P_1 and P_2.

Lemma 2. *Let P^* is a refinement of P. Then*

$$L(P^*, f) \geq L(P, f) \quad \text{and} \quad U(P^*, f) \leq U(P, f).$$

Lemma 3. *For any $f \in \mathcal{B}[a, b]$,*

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$$

Note: Let P^* be a refinement of P. Then, for any $f \in \mathcal{B}[a, b]$,

$$L(P, f) \leq L(P^*, f) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq U(P^*, f) \leq U(P, f).$$

Theorem 1. *A function $f \in \mathcal{B}[a, b]$ is Riemann integrable on $[a, b]$ if and only if for any $\epsilon > 0$, there exists a partition P of $[a, b]$ with $U(P, f) - L(P, f) < \epsilon$.*

Theorem 2. *If a function f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.*

Theorem 3. *If a function f is monotone on $[a, b]$, then f is Riemann integrable on $[a, b]$.*

Remark: For any $x \in [x_{i-1}, x_i]$, let

$$m_i^f = \inf f(x) \qquad m_i^g = \inf g(x)$$
$$m_i^f + m_i^g = \inf f(x) + \inf g(x) \leq \inf(f + g)(x) = m_i^{f+g}$$

Thus,

$$L(P, f) + L(P, g) \leq L(P, f + g)$$

Similarly,

$$M_i^f = \sup f(x) \qquad M_i^g = \sup g(x)$$
$$M_i^f + M_i^g = \sup f(x) + \sup g(x) \geq \sup(f+g)(x) = M_i^{f+g}$$

6.2 Properties of the Riemann Integral

Theorem 4. *Suppose that $f, g \in \mathcal{R}[a,b]$ and $k \in \mathbb{R}$. Then,*

(1) $kf \in \mathcal{R}[a,b]$ *and* $\int_a^b kf = k\int_a^b f$.

(2) $(f+g) \in \mathcal{R}[a,b]$ *and* $\int_a^b f + g = \int_a^b f + \int_a^b g$.

(3) *If* $f, g \in \mathcal{R}[a,b]$, *then* $fg \in \mathcal{R}[a,b]$.

Corollary 1. *Let $f_j \in \mathcal{R}[a,b]$ for $j = 1, \dots, N$. Then,*

$$\sum_{j=1}^N k_j f_j \in \mathcal{R}[a,b] \quad \text{and} \quad \int_a^b \sum_{j=1}^N k_j f_j = \sum_{j=1}^N k_j \int_a^b f_j$$

Theorem 5. *Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b]$ if and only if $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$. In this case,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Theorem 6. *If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ and*

$$\Big|\int_a^b f(x)dx\Big| \leq \int_a^b |f(x)|dx.$$

6.3 Fundamental Theorem of Calculus

Definition 1. *Let f be a function on an interval I. A function F is called an antiderivative of f on I if $F'(x) = f(x)$ for all $x \in I$.*

Remark: An antiderivative, if exists, is NOT unique. If $F'(x) = f(x)$ and $G'(x) = f(x)$ for all $x \in I$, then

$$F(x) = G(x) + C$$

with a constant C.

Theorem 7. (Fundamental Theorem for Calculus) *If $f \in \mathcal{R}[a,b]$ and $F'(x) = f(x)$ on $[a,b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Theorem 8. (Fundamental Theorem for Calculus) *Let $f \in \mathcal{R}[a,b]$ and define*

$$F(x) = \int_a^x f(t)dt.$$

Then the function F is continuous on $[a,b]$. Moreover, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Remark: Integrability of f on $[a,b]$ does NOT imply the existence of an antiderivative of f on $[a,b]$.

Theorem 9. (Mean Value Theoem for Integral) *If f is continuous on $[a,b]$, then there exists $c \in (a,b)$ such that* $\int_a^b f(x)dx = f(c)(b-a)$.

Theorem 10. (Integration by parts) *Let f and g be differentiable on $[a,b]$ with $f', g' \in \mathcal{R}[a,b]$. Then*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'.$$

6.4 Improper Riemann Integrals

Definition 2. *Let $f : [a,b] \to \mathbb{R}$ with $f \in \mathcal{R}[c,b]$ for $c \in (a,b)$. The improper integral of f on $(a,b]$ is defined to be*

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f$$

provided the limit exists. If the limit exists, then the improper integral is said to be convergent. Otherwise, the integral is said to be divergent.
Similarly, we can define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f$$

for the function f which is defined on $[a,b]$ and unbounded at b.

Remark: If $f : [a,b] \to \mathbb{R}$ is unbounded at p, $a < p < b$, we define

$$\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx.$$

Remark: If f is Riemann integrable, then f^2 is also Riemann integrable. However, this is NOT true for the improper Riemann integrals. For example, $f(x) = 1/\sqrt{x}$ for $x \in (0,1]$. Also, if $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$. But, this is NOT true for the improper Riemann integrals.

Definition 3. *Let $f : [a,\infty) \to \mathbb{R}$ with $f \in \mathcal{R}[a,c]$ for $c > a$. The improper integral of f on $[a,\infty)$ is defined to be*

$$\int_a^\infty f(x)dx = \lim_{c \to \infty} \int_a^c f$$

provided the limit exists. If the limit exists, then the improper integral is said to be convergent. Otherwise, the integral is said to be divergent.
Similarly, we can define

$$\int_{-\infty}^a f(x)dx = \lim_{c \to -\infty} \int_c^b f$$

if the limit exists.

Remark: The convergence of the improper integral of f on (a,∞) does not imply the convergence of the improper integral of $|f|$ (See Example 6.4.4 (b)).