Chapter 7 Series of Real Numbers

7.1 Convergence of Tests

Let $\{a_k\}$ be a sequence of real numbers. A series is denoted by $\sum_{k=1}^{\infty} a_k$ and the numbers a_k are called the **terms** of the series. The numbers

$$S_n = \sum_{i=1}^{n} a_k$$

are called the **partial sums** of the series.

Definition : The series $\sum_{k=1}^{\infty} a_k$ converges if $\{S_n\}$ converges to S, i.e., $\lim_{n\to\infty} S_n = S \in \mathbb{R}$ We write

$$S = \sum_{k=1}^{\infty} a_k$$

The number s is called the **sum** of the series. If $\{S_n\}$ is divergent, the series is called divergent

Remark : If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k = 0$.

Theorem 1. (Cauchy's criterion) *A series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|S_n - S_m| = |a_n + a_{n-1} + \cdots + a_{m+1}| < \epsilon$$

for any $n > m \geq N$.

Theorem 2. (The Comparison Test) *Let $0 \leq a_k \leq Mb_k$ for any $k \geq N$.*

(1) *Assume that $\sum b_k$ is convergent, then $\sum a_k$ is also convergent.*
(2) *If $\sum a_k$ is divergent , then $\sum b_k$ is also divergent.*

Corollary 1. *Let $a_k, b_k > 0$ for any $k \in \mathbb{N}$. Assume that $\{a_k/b_k\}$ and $\{a_k/b_k\}$ are both bounded sequences. Then, $\sum a_k$ and $\sum b_k$ are either both converge or both diverge.*

Theorem 3. (Limit Comparison Test) *Let $\sum a_k$ and $\sum b_k$ be two given series with $a_k, b_k 0 \geq 0$ for any $k \in \mathbb{N}$.*

(1) *If*

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L, \quad 0 < L < \infty,$$

then $\sum a_k$ is convergent if and only if $\sum b_k$ is convergent.

(2) *If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum b_k$ is divergent, then $\sum a_k$ is also divergent.*

Remark : If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum a_k$ is convergent, then "Nothing" can be concluded.

Theorem 4. (The Integral Test) *Let $\{a_k\}$ be monotone decreasing sequence and $a_k \geq 0$ for $k \in \mathbb{N}$. Suppose f is a nonnegative and monotonically decreasing function on $[1, \infty)$ satisfying $a_k = f(k)$. Then*

$$\sum_{k=1}^{\infty} a_k < \infty \quad \text{if and only if} \quad \int_1^{\infty} f(x)dx < \infty.$$

Remark (*p*-series) The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. For each $k \in \mathbb{N}$, define

$$a_n = \inf\{s_k : n \geq n\} \quad \text{and} \quad b_n = \sup\{s_k : n \geq n\}$$

Definition Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Then, we define

$$\overline{\lim_{n\to\infty}} s_n = \lim_{n\to\infty} b_n = \inf_{n\in\mathbb{N}} \sup\{s_n : n \geq k\} \quad \textbf{(The limit superior of } \{s_n\}\textbf{)}$$
$$\underline{\lim_{n\to\infty}} s_n = \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \inf\{s_n : n \geq k\} \quad \textbf{(The limit inferior of } \{s_n\}\textbf{)}$$

Remark A real number x is called a **cluster point** of the sequence $\{a_k\}$ if some subsequence of $\{a_k\}$ converges to x, i.e., there exists $\{a_{k_n}\} \subset \{a_k\}$ such that

$$\lim_{n\to\infty} a_{k_n} = x.$$

Let C be the set of all cluster points of $\{a_k\}$. Then, C can be ϕ, a finite set, or an infinite set in $\mathbb{R}$. We write

$$\limsup_{k\to\infty} a_k = \sup C = \text{the largest cluster point}$$
$$\liminf_{k\to\infty} a_k = \inf C = \text{the smallest cluster point.}$$

Theorem 5. (The Ratio Test) *Let $a_k > 0$ and put*

$$\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = R \quad \text{and} \quad \liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = r.$$

(1) *If $R < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.*
(2) *If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.*
(3) *If $r \leq 1 \leq R$, the test is inconclusive.*

Corollary 2. *Let $a_k > 0$ and put $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$. Then,*

(1) *If $L < 1$, then $\sum a_k$ converges.*
(2) *If $L > 1$, then $\sum a_k$ diverges.*

Note: If $\lim a_{k+1}/a_k = 1$, there is NO conclusion.

Theorem 6. (The Root Test) *Let $a_k > 0$ and put*

$$\limsup_{k\to\infty} \sqrt[k]{a_k} = \alpha.$$

Then,

(1) *$\alpha < 1$ implies that $\sum_{k=1}^{\infty} a_k$ converges.*
(2) *$\alpha > 1$ implies that $\sum_{k=1}^{\infty} a_k$ diverges.*
(3) *If $\alpha = 1$, there is NO conclusion.*

Theorem 7. *Let $\{a_k\}$ be a sequence with $a_k \geq 0$.*

$$\underline{\lim_{n\to\infty}} \frac{a_{k+1}}{a_k} \leq \underline{\lim_{n\to\infty}} \frac{a_{k+1}}{a_k} \leq \overline{\lim_{n\to\infty}} \sqrt[k]{a_k} \leq \overline{\lim_{n\to\infty}} \frac{a_{k+1}}{a_k}.$$

Remark If $\sum_{k=1}^{\infty} a_k$ converges by the Ratio test (i.e., $R < 1$), then $\alpha = \overline{\lim_{n\to\infty}} \sqrt[k]{a_k} < 1$. Similarly, if $\sum_{k=1}^{\infty} a_k$ diverges by the Ratio test (i.e., $r > 1$), then $\alpha > 1$

7.2 The Dirichlet Test

Theorem 8. (Abel Partial Summation Formula) *Let $\{a_k\}$ and $\{b_k\}$ be sequences of real numbers. Set $A_0 = 0$ and $A_n = \sum_{k=1}^{n} a_k$ for $n \geq 1$. Then, if $1 \leq p \leq q$,*

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p.$$

Theorem 9. (Dirichlet Test) *Let $\{a_k\}$ and $\{b_k\}$ be sequences of real numbers satisfying the following conditions:*

(1) $A_n = \sum_{k=1}^{n} a_k$ *forms a bounded sequence.*
(2) $b_1 \geq b_2 \geq \cdots \geq 0.$
(3) $\lim_{k\to\infty} b_k = 0.$

Then, $\sum_{k=1}^{\infty} a_k b_k$ converges.

Theorem 10. (The Alternating-Series Test) *Let $\{b_k\}$ be a sequence satisfying the following condition:*

(1) $b_1 \geq b_2 \geq \cdots \geq 0.$
(2) $\lim_{k\to\infty} b_k = 0.$

Then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$ converges.

Theorem 11. *Let $\{b_k\}$ be a sequence satisfying the conditions in Theorem 10. Let*

$$s := \sum_{k=1}^{\infty} (-1)^{k+1} b_k \quad \text{and} \quad s_n := \sum_{k=1}^{n} (-1)^{k+1} b_k.$$

Then, $|s - s_n| \leq b_{n+1}$ for all $n \in \mathbb{N}$.

Theorem 12. (Trigonometric Series) *Suppose $\{b_k\}$ is a sequence satisfying the conditions in Theorem 10. Then*

(1) $\sum_{k=1}^{\infty} b_k \sin kt$ *converges for all $t \in \mathbb{R}$.*
(2) $\sum_{k=1}^{\infty} b_k \cos kt$ *converges for all $t \in \mathbb{R}$ except perhaps $t \neq 2p\pi$ with $p \in \mathbb{N}$.*

7.3 Absolute and Conditional Convergence

The convergence of $\sum_{k=1}^{\infty} a_k$ doesn't imply the convergence of $\sum_{k=1}^{\infty} |a_k|$.

Definition : Let $\{a_k\}$ a sequence of real numbers.

(1) If $\sum_{k=1}^{\infty} |a_k|$ converges, then we say $\sum_{k=1}^{\infty} a_k$ converges absolutely.
(2) If $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges, then we say $\sum_{k=1}^{\infty} a_k$ converges conditionally.

Theorem 13. *If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k$ converges. Moreover, we get*

$$\Big|\sum_{k=1}^{\infty} a_k\Big| \leq \sum_{k=1}^{\infty} |a_k|.$$

Theorem 14. *Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent (to S) and $\sum_{k=1}^{\infty} a'_k$ be an rearrangement of $\sum_{k=1}^{\infty} a_k$. Then*

$$\sum_{k=1}^{\infty} a'_k = S.$$

Theorem 15. (The Ratio and Root Test) *Let*

$$\alpha = \limsup_{k\to\infty} \sqrt[k]{|a_k|}.$$

Also, let

$$\limsup_{k\to\infty} \Big|\frac{a_{k+1}}{a_k}\Big| = R \quad \text{and} \quad \liminf_{k\to\infty} \Big|\frac{a_{k+1}}{a_k}\Big| = r.$$

Then,

1. *If $\alpha < 1$ or $R < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.*
2. *If $\alpha > 1$ or $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.*
3. *If $\alpha = 1$ or $r \le 1 \le R$, there is NO conclusion.*

Definition 1. A series $\sum_{k=1}^{\infty} a'_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$ if there existrs a one-to-one and onto function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $a'_k = a_{\varphi(k)}$.

Theorem 16. (The Ratio and Root Test) *If $\sum_{k=1}^{\infty} a_k$ is a absolutely convergent series and $\sum_{k=1}^{\infty} a'_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$, then*

$$\sum_{k=1}^{\infty} a'_k = \sum_{k=1}^{\infty} a_k$$

Remark: If $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then there exists a rearrangement $\sum_{k=1}^{\infty} a'_k$ which diverges to ∞.