A new method for the analysis of univariate non-uniform subdivision schemes

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Abstract

This paper presents a new method for the analysis of convergence and smoothness of univariate non-uniform subdivision schemes. The analysis involves ideas from the theory of asymptotically equivalent subdivision schemes and non-uniform Laurent polynomial representation together with a new perturbation result. Application of the new method is presented for the analysis of interpolatory subdivision schemes based upon extended Chebyshev systems and for a class of smoothly varying schemes.

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1 Introduction

The analysis of univariate linear uniform stationary subdivision schemes is now well established via the Laurent polynomial tool, see e.g. [5], or the equivalent Joint-Spectral-Radius (JSR) analysis, see e.g. [3, 11]. In this presentation we suggest a new method for the analysis of univariate linear non-uniform subdivision schemes based on a combination of three analysis tools: Asymptotic equivalence of subdivision schemes, Laurent polynomial representation of non-uniform linear schemes and a new result on the convergence of perturbed linear subdivision schemes.

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The schemes analyzed by the new method are defined on uniform grids, by masks which change with the location and the refinement level. As shown in [15], such schemes may still be analyzed using difference schemes, provided the difference schemes exist. The main message of the present work is the possibility of relaxing the necessity of the existence of difference schemes. Instead, certain asymptotic conditions on the subdivision masks are imposed. We demonstrate the application of the new method first to the analysis of $2n$-point interpolatory schemes based upon local interpolation by an extended Chebyshev system. A different approach to the analysis of such schemes is presented in [16], based on specific properties of extended Chebyshev systems. A second application of the new method is to the analysis of a class of non-uniform schemes with masks which vary smoothly along the real line, and which tend to a uniform stationary mask as the level of refinement grows.

Previous analysis of non-uniform subdivision schemes on diadic grids is presented in [1], [10], [13], [14] and [15]. [1] presents necessary and sufficient conditions for the convergence of subdivision schemes induced by a sequence of two-slanted bi-infinite matrices. In [10], sufficient conditions for the convergence of a non-uniform corner cutting and the differentiability of the limit curve are presented. The construction of stationary non-uniform subdivision schemes that have a prescribed approximation power is presented in [13]. The paper [14] investigates the smoothness of quasi-uniform bivariate subdivision. In [15], a Laurent polynomial representation is suggested for the analysis of non-uniform subdivision schemes, and the basic operations required for smoothness analysis are presented. The paper [2] analyzes a 4-point scheme defined on a non-uniform grid.

2 Preliminaries on uniform and non-uniform schemes

A uniform binary stationary subdivision scheme is defined by a fixed mask of coefficients $p = \{p_i\}$ via the recursive relation

$$f_j^k = \sum_{i\in\mathbb{Z}} p_{j-2i} f_i^{k-1} \ , \quad k \geq 1, \quad j \in \mathbb{Z}. \tag{1}$$

Here f_j^k is the value, at refinement level k, attached to the diadic point $j2^{-k}$. A general non-uniform binary subdivision scheme may apply a different mask for any newly defined value. We denote by $p^{j,k}$ the mask of coefficients defining the new value f_j^k at $j2^{-k}$, and the scheme is

$$f_j^k = \sum_{i\in\mathbb{Z}} p_{j-2i}^{j,k} f_i^{k-1} \ , \quad k \geq 1, \quad j \in \mathbb{Z}. \tag{2}$$

Hence, a non-uniform binary subdivision is defined by the set of masks $\{p^{j,k}\}_{k\in\mathbb{Z}_+, j\in\mathbb{Z}}$. In this paper, we assume that all the masks have the same finite support. Note that only odd (even) indices of the mask are used to define a new value at f_j^k for an odd (even) j. Denoting $f^k = \{f_i^k : i \in \mathbb{Z}\}$, another way of representing (2) is

$$f^k = S_k f^{k-1}, \tag{3}$$

where S_k is an operator (infinite matrix). It is further assumed that S_k is a bounded operator, that is,

$$S_k \ : \ \ell^\infty(\mathbb{Z}) \ \to \ \ell^\infty(\mathbb{Z}), \tag{4}$$

and that $f^0 \in \ell^\infty(\mathbb{Z})$.

Definition 2.1. We say that a subdivision scheme is C^m if the sequence of piecewise linear interpolants $\{f^k(\cdot)\}_{k=0}^\infty$,

$$f^k(t) = \sum_j f_j^k B_1(2^k \cdot -j), \tag{5}$$

converges uniformly to a C^m function, denoted by f^∞, where B_1 is the standard hat function on $[-1, 1]$.

Note that the uniform convergence of $\{f^k(\cdot)\}_{k=0}^\infty$ implies that the limit is a C^0 function.

Definition 2.2. A subdivision scheme (3) is termed stable if there exists $M > 0$ such that for all $k, n \in \mathbb{Z}_+$,

$$\|S_{k+n} \cdots S_{k+1} S_k\|_\infty < M. \tag{6}$$

Definition 2.3. A stationary uniform subdivision scheme defined by a mask p is denoted by S_p, and for a converging scheme, we use the notation

$$S_p^\infty f^0 = f^\infty. \tag{7}$$

The basic limit function of S_p is $\phi_p = S_p^\infty\{\delta_{0,i}\}$.

Another notion of stability used in this paper is concerned with the stability of the basis spanned by the integer shifts of the basic limit function:

Definition 2.4. The basic limit function ϕ_p of a stationary uniform subdivision scheme defined by a mask p is termed L_∞-stable if for any bounded sequence $\{a_i\}_{i\in\mathbb{Z}}$,

$$C_1 \sup_{i\in\mathbb{Z}} |a_i| \leq \| \sum_{i\in\mathbb{Z}} a_i \phi_p(x-i)\|_\infty \leq C_2 \sup_{i\in\mathbb{Z}} |a_i|. \tag{8}$$

As shown in [5], it is convenient to present the analysis of uniform subdivision schemes in terms of the Laurent polynomials' representation. A stationary scheme with mask coefficients $p = \{p_i\}$ defines a Laurent polynomial

$$p(z) = \sum_{i\in Z} p_i z^i,$$

and it is convergent if and only if $p(z)/(1+z)$ is a Laurent polynomial, and the scheme it defines is contractive. Furthermore, a scheme is C^m (produces C^m limit functions) if $2^m p(z)/(1+z)^{m+1}$ is a Laurent polynomial, and if the scheme defined by it is convergent. This condition is also necessary for the scheme to be C^m if its basic limit function is L_∞-stable (see e.g. [8, p.114]). The above observations follow from the basic Laurent series relation

$$F_k(z) = p(z)F_{k-1}(z^2), \tag{9}$$

where

$$F_k(z) = \sum_{j \in Z} f_j^k z^j \tag{10}$$

with $\{f_j^k\}$ the values generated at level k by the subdivision scheme. Furthermore, the Laurent series representation of the m-th order divided differences of the values at level k can be represented as

$$D_k^m(z) = 2^{km}(1-z)^m F_k(z) = 2^{km}(1-z)^m p(z)F_{k-1}(z^2). \tag{11}$$

Therefore, the relations (9) and (11), and the fact that $p(z)/(1+z)^m$ is a Laurent polynomial imply the existence of a 'divided difference scheme'

$$D_k^m(z) = 2^m p(z)/(1+z)^m D_{k-1}^m(z^2)\ . \tag{12}$$

The convergence of the m-th order divided difference scheme implies that the original scheme is C^m.

3 Three analysis tools

This work analyzes non-uniform schemes which converge asymptotically to uniform schemes. It suggests some conditions on the asymptotic behavior that are sufficient for deducing smoothness results from the smoothness of the limit uniform scheme. The rudimental tools for the new analysis are described below. They include a new basic result on perturbation in subdivision schemes, asymptotic equivalence analysis, and a Laurent series analysis for the non-uniform case.

3.1 Small perturbations in subdivision schemes

The following perturbation result holds for general non-uniform linear stable subdivision schemes.

Proposition 3.1. *Let $\{S_k\}_{k\geq 0}$ be a linear and stable (C^0) subdivision scheme. Let $\{\epsilon^k\}_{k\geq 0}$ be a sequence of real sequences, $\epsilon^k = \{\epsilon_j^k\}_{j\in\mathbb{Z}}$, satisfying*

$$\sum_{k=0}^{\infty} \|\epsilon^k\|_\infty < \infty.$$

Then the perturbed subdivision scheme

$$f^k = S_{k-1} f^{k-1} + \epsilon^k, \quad k = 1, 2, \dots \,, \tag{13}$$

converges to a C^0 limit for any initial data $f^0 \in \ell^\infty(\mathbb{Z})$.

Proof. Denoting by $S^{[k,\infty)} g^k$ the limit function of the scheme $\{S_j\}_{j \geq k}$ starting at level k with initial values g^k and using the linearity of the scheme in (13), we have

$$\lim_{k \to \infty} \sum_j f_j^k B_1(2^k \cdot -j) = S^{[0,\infty)} f^0 + \sum_{k=1}^{\infty} S^{[k,\infty)} \epsilon^k. \tag{14}$$

By the assumptions of the proposition, all the functions in the infinite sum in the right hand side of (14) are continuous, and the sum is uniformly convergent. Therefore, the scheme (13) defines a continuous limit function. □

For later use, we introduce the notation

$$a_k = \bar{o}(1) \ , \quad as \quad k \to \infty \ , \tag{15}$$

to denote an absolute summable real sequence $\{a_k\}$. More generally, we compare a sequence $\{a_k\}$ with a positive sequence $\{b_k\}$, and denote

$$a_k = \bar{o}(b_k) \ , \quad as \quad k \to \infty \ , \tag{16}$$

if

$$a_k / b_k = \bar{o}(1) \ , \quad as \quad k \to \infty. \tag{17}$$

Proposition 3.1 states that summable perturbations of the values do not affect the convergence of a stable scheme.

3.2 Asymptotic equivalence for non-uniform schemes

Similarly to [7], a non-uniform scheme with the masks $\{p^{j,k}\}$ is said to be asymptotically equivalent to a scheme with the masks $\{\tilde{p}^{j,k}\}$, denoted by $\{p^{j,k}\} \sim \{\tilde{p}^{j,k}\}$, if

$$\sum_{k=0}^{\infty} \sup_{j \in \mathbb{Z}} \{ \| p^{j,k} - \tilde{p}^{j,k} \| \} < \infty, \tag{18}$$

where the norm is any norm on the coefficients of the masks. The results in [7] and [8] refer to uniform non-stationary schemes, i.e., the masks depend only upon the level k. However, the results are easily adaptable to the general non-uniform case:

Proposition 3.2. *If $\{p^{j,k}\} \sim \{\tilde{p}^{j,k}\}$ and the scheme defined by $\{\tilde{p}^{j,k}\}$ is stable and C^0, then so is the scheme defined by $\{p^{j,k}\}$.*

In this paper, we assume that the scheme with the masks $\{p^{j,k}\}$ is asymptotically equivalent to a convergent stationary scheme with the mask p. Therefore, by Proposition 3.2, and since any convergent stationary scheme is stable [7], it immediately follows that the scheme defined by $\{p^{j,k}\}$ is also a C^0 stable scheme. However, the conditions in [7] for higher order smoothness in terms of the masks are quite restrictive and smoothness analysis via "smoothing factors" is developed there. Since we failed to extend this analysis to non-uniform schemes, we suggest in this paper a different method.

3.3 Laurent series formalism for non-uniform schemes

Following [15], we can adapt the Laurent series formalism to the case non-uniform subdivision schemes. Introducing the notation

$$[G(z)]_j = [\sum_{\ell \in \mathbb{Z}} g_\ell z^\ell]_j = g_j, \quad j \in \mathbb{Z}, \tag{19}$$

and recalling the definition of $F_k(z)$ in (10), we present a general linear non-uniform binary subdivision scheme (2) by

$$F_k(z) = \sum_{j \in \mathbb{Z}} [p^{j,k}(z) F_{k-1}(z^2)]_j z^j, \tag{20}$$

where the Laurent polynomials $p^{j,k}(z)$, as in the uniform case, are defined by the corresponding masks,

$$p^{j,k}(z) = \sum_{i \in Z} p_i^{j,k} z^i.$$

The Laurent polynomial $p^{j,k}(z)$ defines the subdivision rule for computing the value f_j^k, which is the value at level k, attached to the parameter value $j2^{-k}$.

As in the uniform case, we denote by $D_k^m(z)$ the Laurent series of the m-th order divided differences at subdivision level k, that is,

$$D_k^m(z) = 2^{km}(1-z)^m F_k(z) = \sum \delta_j^{k,m} z^j,$$

where $\delta_j^{k,m}$ is the m-th divided differences of the values at level k,

$$\delta_j^{k,m} = 2^{mk} \sum_{i=0}^{m} (-1)^i \binom{m}{i} f_{j-i}^k. \tag{21}$$

By (19) and (20), the corresponding Laurent series analogue of (11) in the non-uniform case is

$$
\begin{aligned}
D_k^m(z) &= \sum_{j\in\mathbb{Z}} 2^{km}[(1-z)^m F_k(z)]_j z^j \\
&= \sum_{j\in\mathbb{Z}} 2^{km}\Big[\sum_{i=0}^{m}(-1)^i\binom{m}{i} z^i p^{j-i,k}(z)F_{k-1}(z^2)\Big]_j z^j \\
&= \sum_{j\in\mathbb{Z}} 2^{km}[d_m^{j,k}(z)F_{k-1}(z^2)]_j z^j,
\end{aligned}
\tag{22}
$$

where

$$
d_m^{j,k}(z) \equiv \sum_{i=0}^{m}(-1)^i\binom{m}{i} z^i p^{j-i,k}(z). \tag{23}
$$

We would have liked to proceed further to a relation analogous to (12) with $D_{k-1}^m(z)$ in the right hand side of (22), but this is not possible in the non-uniform case. Instead, we formulate below conditions on the masks $\{p^{j,k}\}$ that lead to a perturbed form of (12), which facilitates the smoothness analysis.

4 The new analysis method

Our new method applies to a subclass of non-uniform subdivision schemes with polynomials $d_m^{j,k}(z)$ in (23) satisfying certain asymptotic conditions.

Definition 4.1. (**Property A**) Consider a non-uniform binary scheme S defined by masks $\{p^{j,k}\}$. We say that the scheme S satisfies **Property A of order m** if

$$
\frac{d^r}{dz^r} d_m^{j,k}(\pm 1) = \bar{o}(2^{-k(m-r)}), \quad as\ k\to\infty. \ \ 0\le r<m. \tag{24}
$$

The main theorem below uses Property A to prove smoothness of non-uniform schemes. Later on, we present and analyze two families of non-uniform schemes which satisfy Property A.

Theorem 4.2. *Consider a non-uniform binary scheme S defined by masks $\{p^{j,k}\}$, satisfying*

$$
\{p^{j,k}\} \sim p, \tag{25}
$$

where p is the mask of a stationary binary scheme S_p. Further assume that S satisfies Property A of orders $1\le \ell \le m$. If S_p is a C^m scheme with a stable basic limit function, then also S is a C^m scheme.

Proof. By Proposition 3.2, it follows that S is convergent and C^0. In order to prove higher smoothness, we examine the convergence of divided differences, using the Laurent polynomials' representation introduced in Section 3.3. Recalling (22), we have

$$
D_k^m(z) = \sum_{j\in\mathbb{Z}} 2^{km}[d_m^{j,k}(z)F_{k-1}(z^2)]_j z^j. \tag{26}
$$

By Property A at $z = 1$,

$$d_m^{j,k}(1) = \bar{o}(2^{-km}), \quad as\ k \to \infty. \tag{27}$$

Hence, using the first order Taylor expansion of $d^{j,k}(z)$ around $z = 1$, we obtain

$$d_m^{j,k}(z) = (1 - z)e^{j,k}(z) + \bar{o}(2^{-km}). \tag{28}$$

From now on, the notation $\bar{o}(b_k)$ also stands for a sequence of Laurent polynomials whose coefficients are $\bar{o}(b_k)$ as $k \to \infty$. Next, by Property A at $z = -1$, it follows that

$$e^{j,k}(-1) = \bar{o}(2^{-km}). \tag{29}$$

Then the Taylor expansions of $e^{j,k}(z)$ around $z = -1$ yields the equation

$$d_m^{j,k}(z) = (1 - z^2)c_1^{j,k}(z) + \bar{o}(2^{-km}). \tag{30}$$

By (24), it turns out that the Laurent polynomial $c_1^{j,k}(z)$ has Property A of order $m - 1$. Hence, we can repeat the above process and get

$$d_m^{j,k}(z) = (1 - z^2)^2 c_2^{j,k}(z) + (1 - z^2)\bar{o}(2^{-k(m-1)}) + \bar{o}(2^{-km}).$$

After m steps of the process, we finally have the equation

$$d_m^{j,k}(z) = (1 - z^2)^m c_m^{j,k}(z) + \sum_{r=0}^{m-1}(1 - z^2)^r \bar{o}(2^{-k(m-r)}). \tag{31}$$

For later use, we denote the last term in (31) as

$$R_m^{j,k}(z) := \sum_{r=0}^{m-1}(1 - z^2)^r \bar{o}(2^{-k(m-r)}). \tag{32}$$

On the other hand, since $\sum_{i=0}^{m}(-1)^i \binom{m}{i} z^i = (1 - z)^m$, it follows from (23) and (25) that

$$d_m^{j,k}(z) = p(z)(1 - z)^m + \bar{o}(1).$$

This, together with (31), yields

$$c_m^{j,k}(z) = \frac{p(z)}{(1 + z)^m} + \bar{o}(1). \tag{33}$$

Plugging (31) in (26) and using (33), we obtain that

$$\begin{aligned} D_k^m(z) &= \sum_{j \in Z}[2^{km}\big((1 - z^2)^m c_m^{j,k}(z) + R_m^{j,k}(z)\big)F_{k-1}(z^2)]_j z^j \\ &= \sum_{j \in Z}\Big[\big(2^{km}\big(\frac{p(z)}{(1 + z)^m} + \bar{o}(1)\big)(1 - z^2)^m + 2^{km}R_m^{j,k}(z)\big)F_{k-1}(z^2)\Big]_j z^j. \end{aligned} \tag{34}$$

Now, we are in a position to represent the relation between the m-th divided differences at level k in (22) and the m-th divided differences at level $k-1$, that is,

$$D_{k-1}^m(z) = \sum_{j \in Z} 2^{(k-1)m}[(1-z)^m F_{k-1}(z)]_j z^j.$$

Specifically, we have

$$D_k^m(z) = \sum_{j \in Z} \big[\big(\frac{2^m p(z)}{(1+z)^m} + \bar{o}(1)\big) D_{k-1}^m(z^2)\big]_j z^j + 2^{km} \sum_{j \in Z} [R_m^{j,k}(z) F_{k-1}(z^2)]_j z^j. \quad (35)$$

We claim that the last term on the right hand side of (35) contributes $\bar{o}(1)$ perturbations, under the assumption that the scheme is C^{m-1}. Indeed, by (32), a typical term in $2^{km} R_m^{j,k}(z) F_{k-1}(z^2)$ is

$$\bar{o}(2^{kr})(1-z^2)^r F_{k-1}(z^2)\ ,\ 0 \le r \le m-1. \quad (36)$$

The values represented by $(1-z^2)^r F_{k-1}(z^2)$ are the rth order divided differences of f^{k-1}, and these are $o(2^{-kr})$ as $k \to \infty$. Therefore, each term of the form (36) is $\bar{o}(1)$. Recalling that p is the generating Laurent polynomial of a C^m scheme with a stable limit function, we know that $2^m p(z)/(1+z)^m$ represents the converging scheme for the mth order divided differences. Hence, the scheme defined by $2^m p(z)/(1+z)^m + \bar{o}(1)$ is convergent as being asymptotically equivalent to a converging scheme, and it is even a stable scheme by Proposition 3.2. Thus, the relation (35) represents a perturbed scheme which satisfies the conditions of Proposition 3.1. We conclude that the m-th order divided differences $\{\delta_j^{k,m}\}$ defined in (21) converge to a continuous limit, hence $\{f_j^k\}$ converge to a C^m limit under the assumption that the limit is C^{m-1}. Using induction, starting with $m=1$, we obtain the claim of the theorem. $\square$

In the following sections, we discuss two examples of families of non-uniform schemes which satisfy Property A.

5 Interpolatory schemes defined by extended Chebyshev systems

In this section, we consider the family of interpolatory schemes defined by extended Chebyshev systems, and show how the new method presented above can be used for analyzing the smoothness of such schemes. The $2n$-point interpolatory subdivision scheme of Dubuc Deslauriers [4] is constructed by demanding that the scheme reproduces polynomials up to degree $2n-1$. Here we discuss the $2n$-point interpolatory scheme reproducing an extended Chebyshev system (ECS), $\Phi_{2n} = \{\phi_m\}_{m=1}^{2n} \subset C^{2n}(I\!R)$. The scheme is in fact defined by replacing the polynomial basis by an ECS Φ_{2n}. A special case of such schemes has been studied in [9], where the ECS is a set of exponential polynomials and the corresponding subdivision schemes are uniform but non-stationary. In general, unlike the polynomial basis, an ECS is not shift or scale invariant. Accordingly, to define the subdivision rule at each new point $s = j2^{-k}$ with j odd at subdivision level k, one has to solve a system of linear equations for the mask coefficients $\{w_i^{j,k}\}$:

$$\sum_{i=-n}^{n-1} w_i^{j,k} \phi_m(s + (2i+1)2^{-k}) = \phi_m(s), \quad 1 \leq m \leq 2n. \tag{37}$$

For $1 \leq m \leq 2n$, let $\psi_m \in span(\{\phi_\ell\}_{\ell=1}^{2n})$ be the function interpolating the function $(\cdot - s)^{m-1}$ and its derivatives up to order $2n-1$ at s. Since Φ_{2n} is an ECS, the functions $\{\psi_m\}_{m=1}^{2n}$ exist, and

$$|\psi_m(x) - (x-s)^{m-1}| \leq C|x-s|^{2n}, \quad 1 \leq m \leq 2n, \tag{38}$$

where the constant $C > 0$ depends on the derivatives of order $2n$ of $\{\phi_\ell\}_{\ell=1}^{2n}$. Replacing ϕ_m by ψ_m in the system (37) and using (38), we obtain an equivalent system for the mask coefficients $\{w_i^{j,k}\}$:

$$\sum_{i=-n}^{n-1} w_i^{j,k}[(2i+1)^m 2^{-mk} + O(2^{-2nk})] = \delta_{m,0} + O(2^{-2nk}), \quad 0 \leq m \leq 2n-1. \tag{39}$$

Multiplying the mth equation in (39) by 2^{mk} with $0 \leq m \leq 2n-1$, we get

$$\sum_{i=-n}^{n-1} w_i^{j,k}[(2i+1)^m + O(2^{(-2n+m)k})] = \delta_{m,0} + O(2^{(-2n+m)k}). \tag{40}$$

Recalling the system defining the mask coefficients of the $2n$-point interpolatory subdivision scheme of Dubuc-Deslauriers, that is,

$$\sum_{i=-n}^{n-1} \tilde{w}_i (2i+1)^m = \delta_{m,0}, \quad 0 \leq m \leq 2n-1, \tag{41}$$

we observe that the system (40) can be viewed as a perturbation of (41). Recall also that the weights $\{w_i^{j,k}\}$ define the subdivision rule at a new point $s = j2^{-k}$, for j odd, at refinement level k. For the Dubuc-Deslauriers scheme those weights are independent upon j and k, i.e., the subdivision is uniform and stationary, and a related Laurent polynomial is defined as

$$\tilde{p}(z) = 1 + \sum_{i=-n}^{n-1} \tilde{w}_i z^{2i+1}. \tag{42}$$

The Laurent polynomials $\{p^{j,k}(z)\}$ for the case of a $2n$-point ECS interpolating subdivision scheme are

$$p^{j,k}(z) = 1 + \sum_{i=-n}^{n-1} w_i^{j,k} z^{2i+1}, \tag{43}$$

where $\{w_i^{j,k}\}$ are obtained by solving (37) with $s = j2^{-k}$. Unlike [15], we do not assume here that the scheme reproduces the constant function, but, as we show below, the scheme $\{p^{j,k}\}$ is asymptotically equivalent to the convergent stationary scheme $S_{\tilde{p}}$ with $\tilde{p}$ defined in (42). Therefore, by [7], it immediately follows that the scheme defined by $\{p^{j,k}(z)\}$ is also a C^0 scheme. However, the tools in [7] for the analysis higher order smoothness are not applicable for the ECS schemes, and we need the analysis developed in this work to prove higher smoothness results.

To show that $\{p^{j,k}\} \sim \tilde{p}$, we view the system (40) as a perturbation of (41), and apply the following perturbation result:

Lemma 5.1. *Let A be a non-singular matrix and let $Ax^{(0)} = b$. Consider the solution $x^{(h)}$ of a perturbed system $(A + hB)x^{(h)} = b + hc$, as $h \to 0$. Then, for a small enough h,*

$$x^{(h)} = x^{(0)} + hy^{(h)}, \quad \text{as } h \to 0, \quad \|y^{(h)}\| \leq M. \tag{44}$$

Proof. For a small enough h, we may represent the inverse of $A + hB$ as

$$\begin{aligned}(A + hB)^{-1} &= A^{-1}(I + hBA^{-1})^{-1} \\ &= A^{-1}\sum_{\ell=1}^{\infty}(hBA^{-1})^{\ell-1} \\ &= A^{-1} + hA^{-1}BA^{-1}\sum_{\ell=1}^{\infty}(hBA^{-1})^{\ell}.\end{aligned}$$

We see that the above infinite sums converge for a sufficiently small h. Therefore, the proof follows by applying the inverse to $b + hc$. $\square$

We now use all the above observations to find the asymptotic behavior of the Laurent polynomials $p^{j,k}(z)$ of the interpolatory schemes reproducing the ECS Φ_{2n}. Using Lemma 5.1 and viewing the system (40) as a perturbation of the system (41) with $h = 2^{-k}$, we obtain the following result.

Proposition 5.2. *Let $\{p^{j,k}(z)\}$ be the Laurent polynomials of the interpolatory scheme reproducing an ECS Φ_{2n}. Then,*

$$p^{j,k}(z) = \tilde{p}(z) + 2^{-k}s^{j,k}(z), \tag{45}$$

where the coefficients of the polynomials $\{s^{j,k}(z)\}$ are uniformly bounded.

A direct consequence of Proposition 5.2 is that $\{p^{j,k}\} \sim \tilde{p}$, from which we conclude that the ECS interpolating scheme is C^0.

The Laurent polynomial $\tilde{p}(z)$ has a root of multiplicity $2n$ at $z = -1$. This property is necessary for applying the Laurent polynomial tool to analyze the smoothness of the

ECS scheme. In order to apply Theorem 4.2 for our non-uniform schemes, we need to study the behavior of $p^{j,k}(z)$ near $z = \pm 1$. Property (45) directly implies that

$$p^{j,k}(-1) = O(2^{-k}), \quad as\ k \to \infty.$$

However, using the system (40), we observe a more delicate behavior of the Laurent polynomial $p^{j,k}(z)$ near $z = \pm 1$.

Proposition 5.3. *Let $\{p^{j,k}(z)\}$ be the Laurent polynomials of the interpolatory scheme reproducing an ECS Φ_{2n}. Then, as $k \to \infty$, we have*

$$\frac{d^m}{dz^m} p^{j,k}(-1) = O(2^{(-2n+m)k}), \quad 0 \le m \le 2n-1, \tag{46}$$

and

$$\frac{d^m}{dz^m} p^{j,k}(1) = 2\delta_{m,0} + O(2^{(-2n+m)k}), \quad 0 \le m \le 2n-1. \tag{47}$$

Proof. By Proposition 5.2, the coefficients $\{w_i^{j,k}\}$ are uniformly bounded. Hence, by (40), we have

$$\sum_{i=-n}^{n-1} w_i^{j,k}(2i+1)^m = \delta_{m,0} + O(2^{(-2n+m)k}), \quad 0 \le m \le 2n-1. \tag{48}$$

Applying the differential operator $L = z\frac{d}{dz}$ to (43) m times with $m > 0$, we have

$$L^m p^{j,k}(z) = \sum_{i=-n}^{n-1} w_i^{j,k}(2i+1)^m z^{2i+1}. \tag{49}$$

Thus it follows, in view of (48), that

$$|L^m p^{j,k}(\pm 1)| = |\sum_{i=-n}^{n-1} w_i^{j,k}(2i+1)^m| = O(2^{(-2n+m)k}), \quad 0 < m \le 2n-1. \tag{50}$$

The results (46) and (47) for $m \neq 0$ follow by recursive use of (50), and (47) for $m = 0$ follows from (48). $\square$

Using the above Proposition 5.3, we are now ready to use Theorem 4.2 to prove the smoothness of interpolatory schemes based on ECSs.

Theorem 5.4. *Let $\Phi_{2n} = \{\phi_i\}_{i=1}^{2n} \subset C^{2n}(\mathbb{R})$ be an ECS for $n \ge 2$, and define the mask of an interpolatory subdivision scheme by (37). Then, the limit functions of the subdivision are converging to a C^m limit, where m is the highest smoothness of the corresponding $2n$-Dubuc-Deslauriers scheme.*

Proof. We have already established that the scheme is convergent, being asymptotically equivalent to the $2n$-Dubuc-Deslauriers scheme. It follows from (46) and (47) that the scheme satisfies Property A of order $m < 2n$. Since the highest smoothness of the corresponding $2n$-Dubuc-Deslauriers scheme is smaller than n [4], we can conclude from Theorem 4.2 that the $2n$-point scheme based upon the ECS is C^m, where m is the highest (integer) smoothness of the corresponding $2n$-Dubuc-Deslauriers scheme. □

Remark 5.1. *Conditions (46) and (47) imply Property A for any $m < 2n$. Relations (46) can hold for any scheme, but relations (47) are limited to the case where the non-uniform scheme is asymptotically equivalent to a stationary interpolatory scheme which reproduces polynomials of degree less than $2n$.*

6 Smoothly varying non-uniform schemes

In this section, we apply the new analysis method of Section 4 to a family of smoothly varying subdivision schemes.

Definition 6.1. (**Smoothly varying masks**) A non-uniform set of masks $\{p^{j,k}\}$ is termed **smoothly varying of order** m if the coefficients of the masks satisfy

$$p_i^{j,k} = \varphi_i(2^{-k}j), \tag{51}$$

where $\varphi_i \in C^m(\mathbb{R})$.

A special class of mth order smoothly varying schemes is the class of schemes with smoothly varying 'smoothing factors'. A factor of the form $(1 + z(1 + c2^{-k}))$ is a smoothing factor. In [7], it is shown that multiplying the Laurent polynomial at refinement level k of a uniform non-stationary scheme by such a factor increases by one the smoothness class of the scheme. This is proved via a corresponding convolution argument. In the case of non-uniform schemes, we cannot use the convolution tool. However, we are able to show a similar result for smoothly varying non-uniform schemes satisfying the following Property B.

Definition 6.2. (**Property B**) A non-uniform binary scheme S defined by masks $\{p^{j,k}\}$ has **Property B of order** m if

$$p^{j,k}(z) = \prod_{\ell=1}^{m+1} \big(1 + z(1 + 2^{-k}g_\ell(2^{-k}j))\big)\big(q(z) + 2^{-k}q^{j,k}(z)\big), \tag{52}$$

where $g_i \in C^m(\mathbb{R})$ and $\{q^{j,k}\}$ is smoothly varying of order m.

Remark 6.1. *It is clear that $\{p^{j,k}\}$ is smoothly varying of order m, and that $\{p^{j,k}\} \sim p$ where $p(z) = (1 + z)^{m+1}q(z)$.*

An important class of schemes with Property B are smoothly varying perturbations of spline subdivision schemes. Indeed, the Lane-Riesenfeld algorithm [12], with non-symmetric repeated binary averages using smoothly varying weights tending to $\frac{1}{2}$ at a rate 2^{-k}, is of the form (52), or more specifically

$$p^{j,k}(z) = (1+z)\prod_{\ell=1}^{m}\big(1+z(1+2^{-k}g_\ell(2^{-k}j))\big)/\big(2+2^{-k}g_\ell(2^{-k}j)\big). \tag{53}$$

Such schemes occur in some geometrical variants of the Lane-Riesenfeld algorithm [6].

First, we show that Property B implies Property A.

Proposition 6.3. *A scheme which satisfies Property B of order m satisfies Property A of order m.*

Proof. For this proof, we need to show the relation in (24). At $z = -1$, we have by (52) that each $p^{j,k}(z)$ satisfies

$$\frac{d^r}{dz^r}p^{j,k}(-1) = \bar{o}(2^{-k(m-r)}), \qquad 0 \le r \le m,$$

as $k \to \infty$. Hence, the equation (24) holds at $z = -1$. Also, to prove that (24) holds at $z = 1$, we start with $r = 0$. By (23), (51) and (52), we obtain that

$$\begin{aligned} d_m^{j,k}(1) &= \sum_{i=0}^{m}(-1)^i\binom{m}{i}p^{j-i,k}(1) \\ &= \Delta^m\big(p(1) + 2^{-k}h(2^{-k}(j-\cdot))\big), \end{aligned}$$

where $h \in C^m(\mathbb{R})$ is a function independent of j and k, and Δ^m is the mth order backward difference operator. Now, $\Delta^m\big(h(2^{-k}(j-\cdot))\big) = O(2^{-km})$ since $h \in C^m(\mathbb{R})$. Therefore, it follows that

$$d_m^{j,k}(1) = O(2^{-(m+1)k}), \quad \textit{as } k \to \infty.$$

For higher values of r, using the form (23), the evaluation of $\frac{d^r}{dz^r}d_m^{j,k}(1)$ involves terms of the form $2^{-k}\Delta^m\big((\cdot)^\ell h_\ell^{(s)}(2^{-k}(j-\cdot))\big)$ with $h_\ell \in C^m(\mathbb{R})$ and $\ell + s \le r$. Since

$$\Delta^m\big((\cdot)^\ell h_\ell^{(s)}(2^{-k}(j-\cdot))\big) = O(2^{-k(m-\ell-s)}), \quad \textit{as } k \to \infty,$$

we obtain

$$\frac{d^r}{dz^r}d_m^{j,k}(1) = \bar{o}(2^{-k(m-r)}), \quad 0 \le r \le m.$$

It finishes the proof. $\square$

Remark 6.2. *The above proposition also holds if the last factor in (52) is replaced by* $\big(q(z) + \bar{o}(1)q^{j,k}(z)\big)$.

Theorem 6.4. *Let S be a non-uniform binary scheme defined by masks $\{p^{j,k}\}$. If S satisfies Property B of order m, and if*

$$\{p^{j,k}\} \sim p, \tag{54}$$

where p is the mask of a stationary binary scheme S_p which is a C^m scheme with a stable basic limit function, then, S is a C^m scheme.

Proof. The proof follows immediately from Proposition 6.3 and Theorem 4.2. □

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