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A non-stationary approximation scheme on scattered centers in $\mathbb{R}^d$ by radial basis functions

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Abstract

A nonstationary approximation scheme on $\mathbb{R}^d$ using scattered translates of a smooth radial basis function (e.g., Gaussian) is developed. The scheme is nonstationary and shown to provide spectral approximation orders, i.e., approximation orders that depend only on the smoothness of the approximands.

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1. Introduction

A great deal of attention has been paid to the area of multivariate approximation of the form

$$s(x) := \sum_{\xi \in X} c_\xi \, \phi(x-\xi), \quad x \in \mathbb{R}^d, \tag{1.1}$$

with ϕ a "suitable" basis function and X a set of arbitrary points in $\mathbb{R}^d$, $d \geqslant 1$ (referred to as "centers"). This approach is known to be effective for approximation to scatter data. The use of a radially symmetric basis function ϕ is particularly useful because (i) it facilitates the evaluation of the approximant; (ii) the accuracy of approximation is usually very satisfactory provided the approximand f is reasonably smooth; (iii) there is enough flexibility in the choice of basis functions. A function ϕ is radial in the sense that $\phi(x) = \Phi(|x|)$ where $|x| := (x_1^2 + \cdots + x_d^2)^{1/2}$. Common choices for ϕ are $\phi(x) = (|x|^2 + c^2)^{m-d/2}$, $m > d/2$, d odd, (multiquadric) and $\phi(x) = \exp(-c|x|^2)$, $c > 0$, (Gaussian).

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Some basis functions ϕ (e.g., multiquadric) are not suitable to be used directly for approximation purposes since they increase polynomially fast around ∞. However, a suitable bell-shaped function

$$\psi(x) = \sum_{\alpha\in\mathbb{Z}^d} \mu(\alpha)\phi(x-\alpha)$$

with the infinite sum being convergent in some topology (e.g., the $C(\mathbb{R}^d)$-topology) is obtained by applying a difference operator to ϕ.

The initial approach to scattered data using radial basis functions has been focused on interpolation at the finite set of scattered points $X \subset \mathbb{R}^d$. The general conditions on ϕ that ensure the existence and uniqueness of the solution of the interpolation problem have been given by Micchelli [11]. Interpolation by translates of a suitable radial basis function is certainly an important approach towards solving the scattered data problem. However, it carries its own disadvantage; as the number of centers increases, one needs to solve a large linear system which is ill-conditioned. Thus, the goal of this paper is to develop another approximation method (other than interpolation) with the properties: (i) it is 'local' in the sense that a coefficient in (1.1) should be determined by a few values of the data, even when many centers are involved in the scheme; (ii) it provides spectral approximation accuracy. Asymptotic approximation properties are usually quantified by the notion of approximation order. In order to make this notion feasible, we measure the 'density' of X (in $\mathbb{R}^d$) by

$$\delta := \delta(X) := \sup_{x\in\mathbb{R}^d} \min_{\xi\in X} |x-\xi|. \tag{1.2}$$

Then, given a sequence $(L_\delta)_{\delta>0}$ of schemes, we say that $(L_\delta)_{\delta>0}$ provides L_∞-*approximation order* $k>0$ if, for every sufficiently smooth $f\in L_\infty(\mathbb{R}^d)$,

$$\|f - L_\delta f\|_{L_\infty(\mathbb{R}^d)} = \mathrm{O}(\delta^k),$$

as δ tends to 0. For more details on radial basis function approximation, the reader is referred to the papers [10,13], and the survey papers [3,5,12].

In considering approximation schemes for discrete data, one observes that there already exist many successful results that address the problem in the case $X=\mathbb{Z}^d$. In contrast, less is known for the general case of X in $\mathbb{R}^d$. In [4,6], quasi-interpolations from radial basis function space with infinitely many centers were studied and both showed that the approximation orders obtained in the scattered case are identical to those that had been known on uniform grids, provided that the schemes are stationary. In particular, Dyn and Ron provide a general tool that allows us to convert *any* known approximation scheme on uniform grids to nonuniform grid, while preserving (to the extent that this is possible) the approximation powers known in the former case. The approach of [6] can be described as follows. Suppose that we are given an approximation scheme on uniform grids

$$f \mapsto \sum_{\alpha\in h\mathbb{Z}^d} \Lambda_h f(\alpha)\phi(\cdot-\alpha).$$

Then, we replace each $\phi(\cdot-\alpha)$ by a suitable combination

$$\phi(\cdot,\alpha) := \sum_{\xi\in X} a(\alpha,\xi)\phi(\cdot-\xi),$$

with X a set of scattered centers we wish to use. As a matter of fact, this method provides a general tool for deriving a scheme for a scattered set X from a scheme on a uniform mesh, instead of approximating the function f directly from the space $S_X(\phi)$, which is defined by

$$S_X(\phi) := \text{closure } S_0(\phi),$$

under the topology of uniform convergence on compact sets, with

$$S_0(\phi) : =\text{span}\{\phi(\cdot - \xi) \colon \xi \in X\},$$

the finite span of $\{\phi(\cdot - \xi) \colon \xi \in X\}$. Since the present state-of-art in the area of approximation on uniform grids is quite satisfactory, it gives hope for finding new approaches into the unyielding scattered case. For this reason, in the present paper, we are concerned with the following goals: first, the conversion tool in [6] is designed to apply to the stationary approximation schemes, but we successfully apply it to the de Boor and Ron's (gridded) nonstationary scheme. In turn, we will observe that the converted scheme provides spectral approximation orders that depend on the smoothness of the function f we approximate. Next, when we convert the gridded scheme to the nonuniform case, we are faced with the issue of choosing the density of the uniform grid, namely, $h\mathbb{Z}^d$, corresponding the scattered center set X: A method for selecting the density h associated with a given X is not given in [6]. Thus, the other goal of this paper is to discuss on the optimal way of selecting the density of uniform grid associated to the given set of scattered points X. Consequently, we obtain an approximation scheme which is independent of any uniform grid issue. In fact, using the conversion method in [6], an approximation scheme on scattered centers [14] was developed by using the 'shifted' surface spline (see Section 4). However, for this scheme, we encountered a heavy numerical integration problem (for the details, see [14]), and hence, a cost effective method has been expected. The scheme presented in this paper is easier to implement than [14], and it can be applicable to any basis function whose Fourier transform is positive (more generally, around the origin).

The following notations are used throughout this paper. When $\mathbf{g}$ is a matrix or a vector, $\|\mathbf{g}\|_p$ indicates its p-norm with $1 \leqslant p \leqslant \infty$. Also, for any $\alpha,\ \beta \in \mathbb{Z}_+^d := \{\gamma \in \mathbb{Z}^d \colon \gamma \geqslant 0\}$, we set

$$\alpha! := \alpha_1! \cdots \alpha_d!, \quad |\alpha|_1 := \sum_{k=1}^{d} \alpha_k, \quad \text{and} \quad \alpha^\beta = \prod_{k=1}^{d} \alpha_k^{\beta_k}.$$

For any $k \in \mathbb{N}$, $\prod_k$ stands for the space of all polynomials of degree $\leqslant k$ in d variables. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t)\, e_{-\theta}(t)\, \mathrm{d}t \qquad e_\theta : x \mapsto \mathrm{e}^{\mathrm{i}\theta \cdot x}.$$

Also, we use the notation $f^\vee$ for the inverse Fourier transform. In particular, the Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$. In this paper, our approximands f may be chosen from the Sobolev space

$$W_p^k(\mathbb{R}^d), \quad 1 \leqslant p \leqslant \infty,\ k \in \mathbb{Z}_+,$$

of all functions whose derivatives of orders $\leqslant k$ are bounded. By $|\cdot|_{k,p}$, we shall denote the homogeneous kth order L^p-Sobolev semi-norm, i.e.,

$$|f|_{k,p} := \sum_{|\alpha|_1 = k} \|D^\alpha f\|_{L_p(\mathbb{R}^d)}.$$

2. Basis functions and the conversion method

We assume through out this study that the function ϕ has a generalized Fourier transform in the sense of tempered distribution, and we require that this distribution coincide on $\mathbb{R}^d\backslash 0$ with some continuous function while having a certain type of singularity (necessarily of finite order) at the origin. Hence, here and in the sequel, we assume that $\hat{\phi}$ satisfies the following properties:

$$|\cdot|^{2m}\hat{\phi} = F > 0, \quad m \geqslant 0, \quad \text{and} \quad F \in L_\infty(\mathbb{R}^d). \tag{2.1}$$

In many cases, we find that the basis function ϕ grows at some polynomial degree away from zero. It may cause to lose local property of the approximation. To circumvent those difficulties, a 'localization process' is necessary. Usually, localization is done by applying a difference operator to ϕ, which constructs a bell-shaped function

$$\psi = \sum_{\alpha\in\mathbb{Z}^d} \mu(\alpha)\phi(\cdot - \alpha) \tag{2.2}$$

The coefficients $\mu : \mathbb{Z}^d \to \mathbb{R}$ are called a *localization sequence*. In our study, μ is assumed to have finite support (generally a milder condition is imposed on μ) and the localized function ψ is assumed to satisfy the condition

$$\sup_x (1 + |x|)^{m_\psi}\psi(x) < \infty, \quad \hat{\psi}(0) \neq 0 \tag{2.3}$$

for some $m_\psi > d$.

Lemma 2.1 (Dyn and Ron [6]). *Assume that $\hat{\phi}$ is continuous on $\mathbb{R}^d\backslash 0$ and has a singularity of order $2m$ at the origin for some positive integer m. Let $(\mu(\alpha))_{\alpha\in\mathbb{Z}^d}$ be the localization sequence in* (2.2) *and assume that the localization ψ satisfies the condition* (2.3). *Assume also that the linear functional*

$$\bar{\mu} : p \mapsto \sum_{\alpha\in\mathbb{Z}^d} \mu(-\alpha)p(\alpha) \tag{2.4}$$

is well defined on $\prod_{2m-1}$. Then $\bar{\mu}$ annihilates $\prod_{2m-1}$.

The conversion method in [6] starts with a known approximation scheme L of the form

$$L : f \to \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot - \alpha)\Lambda f(\alpha)$$

with Λ a bounded operator from $L_\infty(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ into itself. Then one chooses for each shift $\psi(\cdot-\alpha)$ an approximation $\psi(\cdot,\alpha)$ from the space $S_X(\phi)$, and by substituting $\psi(\cdot,\alpha)$ for $\psi(\cdot-\alpha)$, one obtains an approximation of the form

$$L_X : f \to \sum_{\alpha\in\mathbb{Z}^d} \psi(\cdot,\alpha)\Lambda f(\alpha)$$

with Λ as before. The function $\psi(\cdot,\alpha)$ thus lies in $S_X(\phi)$. It is also assumed that $\psi(\cdot,\alpha)$ satisfies the decaying condition in (2.3). It follows that the approximation scheme L_X is a bounded map from $L_\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ into $S_X(\phi)$.

The actual construction of $\psi(\cdot,\alpha)$ is done as follows. We first approximate each $\phi(\cdot-\alpha)$ by a linear combination

$$\phi(\cdot,\alpha) := \sum_{\xi\in X} a(\alpha,\xi)\phi(\cdot-\xi), \tag{2.5}$$

which is called a "pseudo-shift" of ϕ, and then define $\psi(\cdot,\alpha)$ by

$$\psi(\cdot,\alpha) := \sum_{\beta\in\mathbb{Z}^d} \mu(\beta-\alpha)\phi(\cdot,\beta), \tag{2.6}$$

which is a localization of the function $\phi(\cdot,\alpha)$. Under some suitable conditions of ϕ, it is shown in [6] that the scheme L_X provides the same approximation order as L does, provided the scheme is stationary. Thus, in order to extend our discussion to the nonstationary case, we introduce the notion of 'admissible coefficients' $(a(\cdot,\xi))_{\xi\in X}$.

Definition 2.2. A vector $(a(\cdot,\xi))_{\xi\in X}$ is termed ℓ_p-admissible if the following conditions hold:

(a) There exists a constant $c_1>0$ such that for any $\gamma\in\mathbb{R}^d$, $a(\gamma,\xi)=0$ whenever $|\gamma-\xi|>c_1\delta$ with δ the density of X as in (1.2).
(b) The set $\{\mathbf{a}(\gamma) := (a(\gamma,\xi))_{\xi\in X}\colon \gamma\in\mathbb{R}^d\}$ is bounded in $\ell_p(X)$, namely, there exists a constant $c_2>0$ such that $\|\mathbf{a}(\gamma)\|_p \leqslant c_2$ for any $\gamma\in\mathbb{R}^d$.

When, in addition to (a) and (b), it particularly satisfies the polynomial reproduction property

$$\sum_{\xi\in X} a(\gamma,\xi)\,p(\xi) = p, \quad \gamma\in\mathbb{R}^d, \quad p\in\prod\nolimits_n, \tag{2.7}$$

the vector $(a(\gamma,\xi))_{\xi\in X}$ is called ℓ_p-'admissible for $\prod_n$'.

Remark. Assuming that the sequence $(a(\cdot,\xi))_{\xi\in X}$ is admissible for $\prod_n$, we note that the linear system in (2.7) is invariant under the dilation and translation on $\mathbb{R}^d$ and X. Hence, without loss of generality, we assume that the following identity holds in this study:

$$(a(ct,\xi))_{\xi\in X} = (a(t,\xi/c))_{\xi\in X}, \quad c>0. \tag{2.8}$$

For the examples of ℓ_p-admissible vectors $(a(\cdot,\xi))_{\xi\in X}$, the reader is referred to the papers [9] and [13].

3. Approximation schemes and error estimates

When we are looking for an approximant from the space $S_X(\phi)$ with a suitable basis function ϕ in terms of the conversion method discussed earlier, it is essential to choose a good approximation scheme on uniform grid. In the paper [2], de Boor and Ron introduced an optimal approximation scheme from the spaces spanned by shifts of a basis function. One can observe here that spectral approximation order can be obtained if the basis function is smooth and satisfies certain other conditions.

For a given integrable function f, the approximation scheme on uniform grid in [2] is given as follows:

$$L_h : f \to \sum_{\alpha\in\mathbb{Z}^d} \psi_h(\cdot/h-\alpha)\Lambda_h f(\alpha) \tag{3.1}$$

with $\Lambda_h f$ the bounded analytic function

$$\Lambda_h f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{ihx\theta}}{\hat{\psi}_h(h\theta)} m_h(\theta)\hat{f}(\theta)\,d\theta, \tag{3.2}$$

where $m_h^\vee * f$ is a (band-limited) mollification of f. In this case, we may choose

$$\psi_h := \psi(\lambda\cdot), \quad \lambda := \lambda(h).$$

Note that, in terms of the original basis function ϕ_h (i.e., prior to localization), we have the identity

$$\phi_h(x/h-\alpha) = \phi\left(\frac{\lambda}{h}(x-h\alpha)\right).$$

It ensures that the approximant $L_h f$ belongs the space $S_{h\mathbb{Z}^d}(\phi((\lambda/h)\cdot))$. On the other hand, the pseudo-shift $\phi_h(\cdot/h,\alpha)$ corresponding to $\phi_h(\cdot/h-\alpha)$ is defined by

$$\phi_h(\cdot/h,\alpha) = \sum_{\xi\in X} a(h\alpha,\xi)\phi\left(\frac{\lambda}{h}(x-\xi)\right). \tag{3.3}$$

Then, the scattered center variant L_X of L_h is obtained by replacing $\phi_h(\cdot/h-\alpha)$ in L_h by $\phi_h(\cdot/h,\alpha)$ in accordance with the conversion method in [6]:

$$L_X f := \sum_{\alpha\in\mathbb{Z}^d} \psi_h(\cdot/h,\alpha)\Lambda_h f(\alpha). \tag{3.4}$$

From (3.3), it is clear that $L_X f$ belongs to the space $S_X(\phi((\lambda/h)\cdot))$. In particular, one must choose the function ψ_h (indeed, ϕ_h) very carefully so that this approximation schemes are effective (i.e., whose error decay) in some sense.

In this study, we want to approximate functions f in the Sobolev space $W_\infty^k(\mathbb{R}^d)$ which is indeed bigger than the space considered in [2]. Further, in order to get numerically stable scheme on scattered points, we need to adjust the mollification $m_h^\vee * f$ according to the basis function ϕ_h and X. (It is rather different from the case of uniform grid in [2].) Specifically, to apply the scheme L_X to an integrable function $f\in W_\infty^k(\mathbb{R}^d)$, we first mollify f as follows:

$$f^\circ := \sigma\left(\frac{h}{\lambda}\cdot\right)^\vee * f, \tag{3.5}$$

where $\sigma:\mathbb{R}^d\to[0,1]$ is a nonnegative C^∞-cutoff function whose support σ lies in some ball B_η, $\eta>0$; furthermore $\sigma=1$ on $B_{\eta/2}$ and that $\|\sigma\|_\infty=1$. Here and hereafter, we assume that $\hat{\psi}\neq 0$ on

B_η such that $\sigma/\hat{\psi}$ is well defined. Note that λ appears in the definition $\phi_h = \phi(\lambda\cdot)$. It is clear that $f^{\circ} = \sigma((h/\lambda)\cdot)^{\vee} * f$ is band-limited. Then, we apply the scheme L_X to f° instead of f.

Lemma 3.1. *Let $f \in W^k_\infty(\mathbb{R}^d)$ and f° be defined as in* (3.5). *Then, as $h/\lambda \to 0$, we have an estimate of the form*

$$\|f - f^{\circ}\|_{L_p(\mathbb{R}^d)} = \mathrm{o}((h/\lambda)^k), \quad 1 \leqslant p \leqslant \infty.$$

Proof. In this proof, for simplicity, we use the abbreviation $\omega := h/\lambda$. By the basic properties of Fourier transform, it is obvious that $\int_{\mathbb{R}^d} \sigma(\omega\cdot)^{\vee}(\theta)\,\mathrm{d}\theta = 1$ for any $\omega > 0$. Then, taking the Taylor expansion of $f(t-\theta)$ about t, we derive the following identities:

$$\begin{aligned}(f - \sigma(\omega\cdot)^{\vee} * f)(t) &= \int_{\mathbb{R}^d} \sigma(\omega\cdot)^{\vee}(\theta)(f(t) - f(t-\theta))\,\mathrm{d}\theta \\ &= \int_{\mathbb{R}^d} \sum_{0<|v|_1<k} \sigma(\omega\cdot)^{\vee}(\theta)(-\theta)^v \frac{D^v f(t)}{v!} + R_k f(t,\theta)\,\mathrm{d}\theta\end{aligned}$$

with

$$R_k f(t,\theta) := \sum_{|v|_1=k} (-\theta)^v D^v f(t - y\theta)/v!$$

for some $y \in [0,1]$. Then the fact $\int_{\mathbb{R}^d} \sigma(\omega\cdot)^{\vee}(\theta)\theta^{\,v}\,\mathrm{d}\theta = 0$ for any $v \neq 0$ implies that

$$\begin{aligned}(f - \sigma(\omega\cdot)^{\vee} * f)(t) &= \sum_{|v|_1=k} \int_{\mathbb{R}^d} \sigma(\omega\cdot)^{\vee}(\theta)(-\theta)^v D^v f(t - y\theta)\,\mathrm{d}\theta/v! \\ &= \sum_{|v|_1=k} \int_{\mathbb{R}^d} \sigma(\omega\cdot)^{\vee}(\theta)(-\theta)^v (D^v f(t) - D^v f(t - y\theta))\,\mathrm{d}\theta/v! \\ &= (\mathrm{i}\omega)^k \sum_{|v|_1=k} \int_{\mathbb{R}^d} (D^v\sigma)^{\vee}(\theta)(D^v f(t) - D^v f(t - y\omega\theta))\,\mathrm{d}\theta/v!.\end{aligned}$$

One can in fact prove by using Minkowski's inequality that

$$\omega^{-k}\|f - \sigma^{\vee}(\omega\cdot) * f\|_{L_p(\mathbb{R}^d)} \leqslant C \sum_{|v|_1=k} \int_{\mathbb{R}^d} |(D^v\sigma)^{\vee}(\theta)|\,\|D^v f - D^v f(\cdot - y\omega\theta)\|_{L_p(\mathbb{R}^d)}\,\mathrm{d}\theta$$

with $1 \leqslant p \leqslant \infty$ and $y \in [0,1]$. It is known (see [7]) that $\|D^\beta f(\cdot - y\omega\theta) - D^\beta f\|_{L_2(\mathbb{R}^d)} \to 0$, for any $\theta \in \mathbb{R}^d$, as $\omega \to 0$. Therefore, by applying the Lebesgue Dominated Convergence Theorem, we have the convergence property $\|f - \sigma(\omega\cdot)^{\vee} * f\|_{L_p(\mathbb{R}^d)} = \mathrm{o}(\omega^k)$ as ω tends to 0. □

From [6], we find that

Lemma 3.2. *Let L_h be defined as in* (3.1). *For any function $f \in W^k_\infty(\mathbb{R}^d)$, f^{o} is defined as in* (3.5). *Then, we have an error bound of the form*

$$\|f^{\mathrm{o}} - L_h f\|_{L\infty(\mathbb{R}^d)} \leqslant Ch^k |f|_{0,\infty}$$

with C depending on k.

The next lemma is useful to estimate $(L_h - L_X)f$.

Lemma 3.3. *Let ϕ be a smooth basis function, and assume that $\hat{\phi}$ is continuous on $\mathbb{R}^d \backslash 0$ and has a singularity of order $2m$ at the origin for some nonnegative integer m. Assume that the coefficients $(a(\alpha,\xi))_{\xi\in X}$* for *$\phi(\cdot,\alpha)$ is admissible for $\prod_{n-1}$ with $n > 2m+1$. Then*

$$|\phi_h(x/h-\alpha) - \phi_h(x/h,\alpha)| \leqslant C\lambda^{n-d-1}\left(\frac{\delta}{h}\right)^n (1+|x/h-\alpha|)^{-d-1},$$

where C is independent of X, x, h, and α.

Proof. Since $\|g\|_{L_\infty(\mathbb{R}^d)} \leqslant \|\hat{g}\|_{L_1(\mathbb{R}^d)}$, it is sufficient to prove that the Fourier transform of $(1+|\cdot|)^{d+1}(\phi_h(\cdot) - \phi_h(\cdot+\alpha,\alpha))$ is bounded by $C\lambda^{n-d-1}(\delta/h)^n$ in L_1-norm, where the pseudo-shift $\phi_h(\cdot+\alpha,\alpha)$ can be written as

$$\phi_h(\cdot+\alpha,\alpha) = \sum_{\xi\in X} a(\alpha,\xi/h)\phi_h(\cdot+\alpha-\xi/h)$$

(see (2.8) and (3.3)). This is equivalent to show that, for any $\gamma \in \mathbb{Z}^d_+$ with $|\gamma|_1 \leqslant d+1$, the Fourier transform of $(\cdot)^\gamma(\phi_h(\cdot) - \phi_h(\cdot+\alpha,\alpha))$ is bounded by $C\lambda^{n-d-1}(\delta/h)^n$ in L_1-norm. Note that

$$[(\cdot)^\gamma(\phi_h(\cdot) - \phi_h(\cdot+\alpha,\alpha))]^\wedge(\theta) = \mathrm{i}^{|\gamma|_1} D^\gamma[\hat{\phi}_h(\theta)(1-E_\alpha(\theta))],$$

where

$$E_\alpha(\theta) = \sum_{\xi\in X} a(\alpha,\xi/h)\mathrm{e}^{\mathrm{i}\theta\cdot(\alpha-\xi/h)}.$$

Hence, by using Leibniz' rule, we will show that

$$\int_{\mathbb{R}^d} |D^{\gamma-\beta}\hat{\phi}_h(\theta) D^\beta(1-E_\alpha(\theta))|\,\mathrm{d}\theta \leqslant C\lambda^{n-d-1}\left(\frac{\delta}{\lambda}\right)^n. \tag{3.6}$$

First, assume that $\beta > 0$ and let $P_0(x)$ be the Taylor polynomial of degree $n - |\beta|_1 - 1$ of the function e^x around the origin. Using the polynomial reproduction property of $(a(\cdot,\xi/h))_{\xi\in X}$, it is easy to verify the following relations:

$$\begin{aligned} D^\beta(1-E_\alpha(\theta)) &= \mathrm{i}^{|\beta|_1} \sum_{\xi\in X} a(\alpha,\xi/h)(\alpha-\xi/h)^\beta \mathrm{e}^{\mathrm{i}\theta\cdot(\alpha-\xi/h)} \\ &= \mathrm{i}^{|\beta|_1} \sum_{\xi\in X} a(\alpha,\xi/h)(\alpha-\xi/h)^\beta P_0(\mathrm{i}\theta\cdot(\alpha-\xi/h)) \end{aligned}$$

$$+\mathrm{i}^n \sum_{\xi\in X} a(\alpha,\xi/h)(\alpha-\xi/h)^\beta(\theta\cdot(\alpha-\xi/h))^{n-|\beta|_1}\mathrm{e}^{\mathrm{i}y\theta\cdot(\alpha-\xi/h)}$$

$$=\mathrm{i}^n \sum_{\xi\in X} a(\alpha,\xi/h)(\alpha-\xi/h)^\beta\theta\cdot(\alpha-\xi/h)^{n-|\beta|_1}\mathrm{e}^{\mathrm{i}y\theta\cdot(\alpha-\xi/h)}$$

for some y between the origin and $\mathrm{i}\theta\cdot(\alpha-\xi/h)$. From the property (a) in Definition 2.2, that is, $a(t,\xi)=0$ whenever $|\alpha-\xi/h|\geqslant c(\delta/h)$ with δ in (1.2), we obtain the bound

$$D^\beta(1-E_\alpha)(\theta)\leqslant C(\delta/h)^n|\theta|^{n-|\beta|_1}.$$

In a similar fashion, we can prove this inequality for the case $\beta=0$. Thus, invoking the definition $\phi_h=\phi(\lambda\cdot)$ and the relation $|\gamma|_1\leqslant d+1$, we get

$$\int_{\mathbb{R}^d} D^{\gamma-\beta}\hat{\phi}_h(\theta)D^\beta(1-E_\alpha(\theta))\,\mathrm{d}\theta\leqslant C(\delta/h)^n\int_{\mathbb{R}^d}|\theta|^{n-|\beta|_1}D^{\gamma-\beta}\hat{\phi}(\theta/\lambda)\,\mathrm{d}\theta/\lambda^d$$

$$\leqslant C(\delta/h)^n\lambda^{n-d-1}\int_{\mathbb{R}^d}|\theta|^{n-|\beta|_1}D^{\gamma-\beta}\hat{\phi}(\theta)\,\mathrm{d}\theta.$$

Finally, in order to complete the proof, we need to show that the last integration (above) makes sense. Since $D^{\gamma-\beta}\hat{\phi}$ decays fast around ∞, the function $|\cdot|^{n-|\beta|_1}D^{\gamma-\beta}\hat{\phi}$ is in $L_1(N_\infty)$ for some neighborhood N_∞ of ∞. Also, we see that the distribution $D^{\gamma-\beta}\hat{\phi}$ has a singularity of order $2m+|\gamma-\beta|_1$ at the origin. Thus, we find that the function $|\cdot|^{n-|\beta|_1}D^{\gamma-\beta}\hat{\phi}$ has a singularity of order $2m+|\gamma|_1-n$ and $2m+|\gamma|_1-n<d$ by the conditions $n>2m+1$ and $|\gamma|_1\leqslant d+1$. It implies that $|\cdot|^{n-|\beta|_1}D^{\gamma-\beta}\hat{\phi}$ is in $L_1(N_0)$ with N_0 a neighborhood at the origin. □

Lemma 3.4. *Let L_h and L_X be the schemes defined as in* (3.1) *and* (3.4), *respectively. Assume that $\hat{\phi}$ is continuous on $\mathbb{R}^d\backslash 0$ and has a singularity of order $2m$ at the origin for some nonnegative integer m. Assume further that the coefficients $(a(\alpha,\xi))_{\xi\in X}$ for $\phi(\cdot,\alpha)$ are admissible for $\prod_{n-1}$. Then, for every function $f\in W^k_\infty(\mathbb{R}^d)$, we have the error bound*

$$\|(L_h-L_X)f\|_{L_\infty(\mathbb{R}^d)}\leqslant C\left(\frac{\delta}{h}\right)^n\lambda^{n-1}h^{\min(k,2m)}|f|_{p,\infty}$$

with C independent of X and h.

Proof. First, from the definitions of L_h and L_X, we can write $(L_h-L_X)f(x)$ as follows:

$$(L_hf-L_Xf)(x)=\sum_{\alpha\in\mathbb{Z}^d}(\psi_h(x/h-\alpha)-\psi_h(x,\alpha))\Lambda_hf(\alpha)$$

$$=\sum_{\beta\in\mathbb{Z}^d}\sum_{\alpha\in\mathbb{Z}^d}(\phi_h(x/h-\alpha)-\phi_h(x/h,\alpha))\mu(\alpha-\beta)\Lambda_hf(\beta).$$

Recalling the definition $\phi_h = \phi(\lambda\cdot)$, due to Lemma 3.3, the above double sum converges absolutely, and summation by parts implies that

$$
\begin{aligned}
(L_h f - L_X f)(x) &= \sum_{\alpha\in\mathbb{Z}^d} (\phi_h(x/h-\alpha) - \phi_h(x/h,\alpha)) \sum_{\beta\in\mathbb{Z}^d} \mu(\alpha-\beta)(\Lambda_h f)(\beta) \\
&\leqslant C\lambda^{n-d-1}\left(\frac{\delta}{h}\right)^n \sum_{\alpha\in\mathbb{Z}^d} |1+(x/h-\alpha)|^{-d-1} \sum_{\beta\in\mathbb{Z}^d} \mu(-\beta)\Lambda_h f(\alpha+\beta),
\end{aligned}
\tag{3.7}
$$

the inequality being a consequence of Lemma 3.3. According to the definition of the linear functional $\bar{\mu}$ in (2.4), we have the identity

$$
\bar{\mu}(\Lambda_h f(\alpha+\cdot)) = \sum_{\beta\in\mathbb{Z}^d} \mu(-\beta)\Lambda_h f(\alpha+\beta).
$$

Let $p := \min(k, 2m)$, and let $T_\alpha(\Lambda_h f)$ be the Taylor polynomial of degree $p-1$ of $\Lambda_h f$ about $\vartheta = \alpha$. Then, the fact that $\bar{\mu}$ annihilates $\prod_{p-1}$ (see Lemma 2.1) implies that

$$
\bar{\mu}(T_\alpha(\Lambda_h f)(\alpha+\cdot)) = \sum_{\beta\in\mathbb{Z}^d} \mu(-\beta)\, T_\alpha(\Lambda_h f)(\alpha+\beta) = 0.
$$

Therefore, we have

$$
\sum_{\beta\in\mathbb{Z}^d} \mu(-\beta)(\Lambda_h f)(\alpha+\beta) = \sum_{|v|_1=p} \sum_{\beta\in\mathbb{Z}^d} \mu(-\beta)\, \frac{\beta^v}{v!}\, D^v(\Lambda_h f)(\beta_\alpha)
\tag{3.8}
$$

with β_α between α and $\alpha+\beta$. Now, using the relation $\hat{\psi}_h(h\cdot) = \hat{\psi}((h/\lambda)\cdot)\lambda^{-d}$, from the explicit formula of $\Lambda_h f$ in (3.2), we get the expression

$$
\begin{aligned}
D^v(\Lambda_h f)(x) &= (\mathrm{i}h)^p \lambda^d (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\sigma}{\hat{\psi}}(h\theta/\lambda)\theta^{\,v} \hat{f}(\theta)\mathrm{e}^{\mathrm{i}hx\cdot\theta}\,\mathrm{d}\theta \\
&= (\mathrm{i}h)^p \lambda^d \left[\left(\frac{\sigma}{\hat{\psi}}\left(\frac{h}{\lambda}\cdot\right)\right)^{\vee} * ((\cdot)^v \hat{f})^{\vee}\right](hx),
\end{aligned}
$$

where $|v|_1 = p$. Thus, it clearly follows from (3.8) that

$$
\left|\sum_{\beta\in\mathbb{Z}^d} \mu(-\beta)(\Lambda_h f)(\alpha+\beta)\right| \leqslant Ch^p\lambda^d |f|_{p,\infty}
\tag{3.9}
$$

with a constant C independent of h, λ, and f, where $p = \min(k, 2m)$. Consequently, combining the relations (3.7) and (3.9), we obtain the required results. □

Now, we are ready provide an approximation scheme on scattered centers which is independent of density of the uniform grid argument. On the base of the above results, we choose $h := \delta$ and

$\lambda := \delta^{1-r}$ with $r \in (0,1)$. Then, the final version of our approximation scheme is defined by

$$L_X f := \sum_{\alpha\in\mathbb{Z}^d} \psi_\delta(\cdot/\delta,\alpha)\Lambda_\delta f(\alpha), \tag{3.10}$$

where $\psi_\delta = \psi(\delta^{1-r}\cdot)$ and

$$\Lambda_\delta f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\sigma(\delta^r\theta)}{\hat{\psi}_\delta(\delta\theta)} \hat{f}(\theta) e^{i\delta x\cdot\theta}\, d\theta.$$

The main result of this section is as follows:

Theorem 3.4. *Assume that $\hat{\phi}$ is continuous on $\mathbb{R}^d\backslash 0$ and has a singularity of order $2m$ at the origin for some nonnegative integer m. Also assume that the sequence $(a(\cdot,\xi))_{\xi\in X}$ for $\phi(\cdot,\alpha)$ be admissible for $\prod_{n-1}$ with $n > 2m+1$. Then, for every $f \in W_\infty^k(\mathbb{R}^d)$, the approximation scheme L_X in* (3.10) *satisfies the following error estimate*:

$$\|f - L_X f\|_{L_\infty(\mathbb{R}^d)} = o(\delta^{rk}) + \begin{cases} O(\delta^{(1-r)n+2m}) & \text{if } k \geqslant 2m, \\ o(\delta^{(1-r)n+k}) & \text{if } k < 2m. \end{cases}$$

Proof. It is useful to divide the error $(f - L_X f)(x)$ as follows:

$$(f - L_X f)(x) = (f - f^{\circ})(x) + (f^{\circ} - L_h f)(x) + (L_h f - L_X f)(x)$$

Then, this theorem is proved immediately by Lemmas 3.1, 3.2, and 3.4. □

Corollary 3.5. *Under the same conditions and notations as in the Theorem* 3.4, *assume that $f \in W^\infty(\mathbb{R}^d)$ with $k \geqslant 2m$. Let the number n $(> 2m+1)$ be chosen to satisfy the condition $(1-r)n + 2m > rk$ for a given $r \in (0,1)$. Then, we have*

$$\|f - L_X f\|_{L(\mathbb{R}^d)} = o(h^{rk}).$$

Remark. The definition of the pseudo-shift $\phi_\delta(\cdot,\alpha)$, $\alpha \in \mathbb{Z}^d$, in (3.3) leads to the explicit form

$$L_X f(x) := \sum_{\alpha\in\mathbb{Z}^d} \psi_\delta(\cdot/\delta,\alpha)\Lambda_\delta f(\alpha) = \sum_{\xi\in X} c_\xi(f)\phi\left(\frac{\lambda}{\delta}(x-\xi)\right),$$

where

$$c_\xi(f) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\in\mathbb{Z}^d} a(\delta(\alpha+\beta),\xi)\mu(\beta)\Lambda_\delta f(\alpha).$$

It is clear that $L_X f$ is an element of $S_X(\phi((\lambda/\delta)\cdot))$.

4. Examples

We discuss here several examples by which spectral approximation orders are obtained.

Example 4.1. The Gaussian kernel: $\phi(x) = \exp(-|x|^2/4)$. Its Fourier transform is of the form $\hat{\phi} = c\mathrm{e}^{-|\cdot|^2}$, $c > 0$, which vanishes nowhere. According to Theorem 3.4, we choose $\lambda(\delta) = \delta^{1-r}$ with $r \in (0,1)$. Then the approximant $L_X f$ is from the space $S_X(\phi(\delta^{-r}\cdot))$ with $\phi(\delta^{-r}\cdot) = \mathrm{e}^{-\delta^{-2r}|\cdot|^2}$ and we can get approximation order $\mathrm{o}(\delta^{rk})$.

Example 4.2. Let the radial basis function ϕ be chosen to be one of what follows:

(a) $\phi_c(x) := (-1)^{\lceil m-d/2 \rceil}(|x|^2 + c^2)^{m-d/2}$, d odd, $m > d/2$ (multiquadrics),
(b) $\phi_c(x) := (-1)^{m-d/2+1}(|x|^2+c^2)^{m-d/2}\log(|x|^2+c^2)^{1/2}$, $m > d/2$, d even ('shifted' surface splines).
(c) $\phi_c(x) := (|x|^2 + c^2)^{m-d/2}$, $0 < m < d/2$ (inverse multiquadrics),

where d, $m \in \mathbb{N} := \{1,2,\ldots\}$ and $c > 0$, and where $\lceil s \rceil$ indicates the smallest integer greater than s. Note that we stress the parameter c by using the notation ϕ_c. When $c = 0$ in the case of (b), the function ϕ_0 is the so-called surface spline. The properties of these basis functions are quite well understood, both theoretically as well as practically. We find (see [8]) that the Fourier transform of ϕ_c is of the form

$$\hat{\phi}_c = c(m,d)\tilde{K}_{m/2}(c\cdot)|\cdot|^{-2m}$$

where $c(m,d)$ is a positive constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|) \neq 0$, $t \geqslant 0$, with $K_\nu(|t|)$ the modified Bessel function of order ν. It is well known from literature (e.g., [1]) that

$$\tilde{K}_\nu \sim (1 + |\cdot|^{(2\nu-1)/2}) \exp(-|\cdot|).$$

Due to Theorem 3.4, we choose $\lambda(\delta) = \delta^{1-r}$ with $r \in (0,1)$. Then the approximant $L_X f$ is from the space $S_X(\phi_{\delta^r})$, and we can get approximation order $\mathrm{o}(\delta^{rk})$.

Remark. By using the 'shifted' surface spline (see Example 4.2 (b)), a nonstationary approximation scheme on scattered centers is introduced in [14]. It is independent of the issue of choosing the density of the uniform grid $h\mathbb{Z}^d$, and it can be written in the form of integral as follows:

$$\int_{\mathbb{R}^d} \psi_c(x/\omega, t)\bar{\Lambda}_\omega f(\omega t)\,\mathrm{d}t, \quad \omega = \omega(\delta), \tag{4.1}$$

where $\bar{\Lambda}_\omega$ is the operator

$$\bar{\Lambda}_\omega f(x) = \int_{\mathbb{R}^d} \frac{\sigma(\omega\theta)}{\hat{\psi}_c(\omega\theta)} \hat{f}(\theta)\mathrm{e}^{\mathrm{i}\omega x\cdot\theta}\,\mathrm{d}\theta.$$

In practice, for this scheme (4.1), we encounter a numerical integration problem and a cost effective way for computation should be addressed. However, we will see that the scheme (3.10) in this study can be interpreted as a discretization of the above integral, while preserving spectral approximation orders. For the proof of this claim, recalling the definition $\psi_{c,\delta} = \psi_c(\lambda\cdot)$ with $\lambda = \lambda(\delta)$, we see that

$$\hat{\psi}_{c,\delta}(\delta\cdot) = \lambda^{-d}\hat{\psi}_c\left(\frac{\delta}{\lambda}\cdot\right).$$

Denote $\omega := \delta/\lambda$. It is easy to show that

$$\Lambda_\delta f = \lambda^d \bar{\Lambda}_\omega f(\lambda\alpha).$$

Then, it follows that

$$\sum_{\alpha\in\mathbb{Z}^d}\psi_{c,\delta}(x/\delta,\alpha)\Lambda_\delta f(\delta\alpha).=\sum_{\alpha\in\mathbb{Z}^d}\lambda^d\psi_c(x/\omega-\lambda\alpha))\bar{\Lambda}_\omega f(\lambda\alpha).$$

The right-hand side can be amounted as the Riemann sum approximation to (4.1). Thus, the claim is justified. In addition, as we discussed in the previous section, the primary concern of the present work is to discuss on the optimal way of the conversion method for the nonstationary case. In the near future, some suitable algorithm will be developed along with computational results with different basis functions.

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