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L_p-ERROR ESTIMATES FOR "SHIFTED" SURFACE SPLINE INTERPOLATION ON SOBOLEV SPACE

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ABSTRACT. The accuracy of interpolation by a radial basis function ϕ is usually very satisfactory provided that the approximant f is reasonably smooth. However, for functions which have smoothness below a certain order associated with the basis function ϕ, no approximation power has yet been established. Hence, the purpose of this study is to discuss the L_p-approximation order ($1 \leq p \leq \infty$) of interpolation to functions in the Sobolev space $W_p^k(\Omega)$ with $k > \max(0, d/2 - d/p)$. We are particularly interested in using the "shifted" surface spline, which actually includes the cases of the multiquadric and the surface spline. Moreover, we show that the accuracy of the interpolation method can be at least doubled when additional smoothness requirements and boundary conditions are met.

1. Introduction

Radial basis functions provide a convenient and simple tool to reconstruct multivariate functions from scattered data. Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, and let $X := \{x_1, \ldots, x_N\}$ be a discrete set in Ω. Let Π_m denote the subspace of $C(\mathbb{R}^d)$ consisting of all d-variate algebraic polynomials of degree less than m. Radial basis function interpolation to a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ on a set X starts with choosing a function $\phi : \mathbb{R}^d \to \mathbb{R}$ and defining an interpolant by

$$s_{f,X}(x) := \sum_{i=1}^{\ell} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi(x - x_j), \tag{1.1}$$

where $p_1, \ldots, p_\ell$ is a basis of Π_m and α_j $(j = 1, \ldots, N)$ are chosen so that

$$\sum_{j=1}^{N} \alpha_j p_i(x_j) = 0, \quad 1 \leq i \leq \ell. \tag{1.2}$$

For a wide choice of functions ϕ and polynomial orders m, including the case $m = 0$, the coefficients of $s_{f,X}$ are required to satisfy the $(N+\ell) \times (N+\ell)$ system of linear equations, which can be written in matrix form as

$$\begin{pmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^{\mathbf{T}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

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where $\mathbf{A}$ and $\mathbf{P}$ are the $N \times N$ and $\ell \times \ell$ matrices that have the elements $\mathbf{A}_{ij} = \phi(x_i - x_j)$ and $\mathbf{P}_{ij} = p_i(x_j)$. Further, $\mathbf{a} \in \mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}^\ell$ are the vectors of coefficients of $s_{f,X}$, and the components of $\mathbf{f}$ are the data $f(x_j)$, $j = 1, \ldots, N$. The general conditions on ϕ that ensure the nonsingularity of the above system have been given by Micchelli [M]. The function ϕ is radial in the sense that $\phi(x) = \Phi(|x|)$, where $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$, and we assume $\phi = \Phi(|\cdot|)$ to be strictly conditionally positive definite of order m, which implies that the matrix $\mathbf{A}$ is positive definite on the subset of vectors $u \in \mathbb{R}^N$ satisfying $\sum_{j=1}^N u_j p(x_j) = 0$ with $p \in \Pi_m$. For $m > 0$, we require X to have the nondegeneracy property for Π_m, i.e., any polynomial in Π_m which vanishes on X must be identically zero. For more details, the reader is referred to the papers [Du], [MN1], [MN2], [WS], and the survey papers [D], [Bu1] and [P1].

We use the following notation throughout this paper. When $\mathbf{g}$ is a matrix or a vector, $\|\mathbf{g}\|_p$ indicates its p-norm with $1 \le p \le \infty$. For $\alpha, \beta \in \mathbb{Z}_+^d := \{(\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d : \gamma_k \ge 0\}$, we set $\alpha! := \alpha_1! \cdots \alpha_d!$, $|\alpha|_1 := \sum_{k=1}^d \alpha_k$, and $\alpha^\beta = \alpha_1^{\beta_1} \cdots \alpha_d^{\beta_d}$. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t) \exp(-i\theta \cdot t)\, dt.$$

Also, for a function $f \in L_1(\mathbb{R}^d)$, we use the notation $f^\vee$ for the inverse Fourier transform. The Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$.

In this paper, we are particularly interested in the basis function that is obtained from the fundamental solution of the iterated Laplacian by the shifting $|x| \mapsto (|x|^2 + \lambda^2)^{1/2}$ with $\lambda \ge 0$,

$$\phi_\lambda(x) := \begin{cases} (-1)^{\lceil m-d/2 \rceil} (|x|^2 + \lambda^2)^{m-d/2}, & d \text{ odd}, \\ (-1)^{m-d/2+1} (|x|^2 + \lambda^2)^{m-d/2} \log(|x|^2 + \lambda^2)^{1/2}, & d \text{ even}, \end{cases} \tag{1.3}$$

where d, $m \in \mathbb{N} := \{1, 2, \ldots\}$, $m > d/2$, and where $\lceil s \rceil$ indicates the smallest integer greater than s. This function ϕ_λ is called the "shifted" surface spline function. When d is odd, ϕ_λ is called the multiquadric, and when $\lambda = 0$, the function ϕ_0 is the so-called surface spline. The interpolation properties of the "shifted" surface spline interpolation have been studied in many articles. The reader is referred to the papers cited above.

We demand some hypotheses on the domain Ω over which the error between f and $s_{f,X}$ is measured. These assumptions are listed as follows:

(a) $\Omega \subset \mathbb{R}^d$ is an open bounded domain with a Lipschitz boundary.
(b) Ω has the cone property.

In order to discuss the extent to which $s_{f,X}$ approximates f, we define the "density" of X in Ω to be the number

$$h := h(X; \Omega) := \sup_{x \in \Omega} \min_{x_j \in X} |x - x_j|. \tag{1.4}$$

Actually, in this study, we need a stability result on the interpolation process. Therefore, we define the separation distance within X by

$$q := \min_{1 \le i \ne j \le N} |x_i - x_j|/2. \tag{1.5}$$

Here and in the rest of the paper, we assume, without great loss, that there exists a constant $\eta > 0$, independent of X, such that

$$h/q \leq \eta. \tag{1.6}$$

This condition asserts that the number of the scattered points in the set X is bounded by ch^{-d}, i.e., $N \leq ch^{-d}$, where the constant c is independent of X.

The accuracy of the aforesaid interpolation method is usually very satisfactory provided that the approximand f itself is reasonably smooth. Indeed, most of the current studies of radial basis function interpolation estimate errors for a class of functions f whose Fourier transforms are dominated by the Fourier transform $\hat{\phi}_\lambda$ in the sense that

$$\int_{\mathbb{R}^d} |\hat{f}(\theta)|^2 \hat{\phi}_\lambda^{-1}(\theta) d\theta < \infty. \tag{1.7}$$

In this case, an approximand f is required to have a certain smoothness associated with the expression (1.7). Specifically, in the case $\lambda = 0$, f should have a smoothness of order m. Even worse, when $\lambda > 0$, the Fourier transform $\hat{\phi}_\lambda$ decays exponentially at infinity. Hence, the approximands need to be extremely smooth for an effective error analysis. Unfortunately, no convergence order for functions which are less smooth has been provided yet. For this reason, in the present paper, we are concerned with providing the L_p-approximation order ($1 \leq p \leq \infty$) of interpolation to functions f in a Sobolev space. For $k \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$, the Sobolev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) := \{f : \|f\|_{k,L_p(\Omega)} := \big(\sum_{|\alpha|_1 \leq k} \|D^\alpha f\|_{L_p(\Omega)}^p \big)^{1/p} < \infty\}$$

with the usual modification when $p = \infty$, namely,

$$\|f\|_{k,L_\infty(\Omega)} := \sum_{|\alpha|_1 \leq k} \|D^\alpha f\|_{L_\infty(\Omega)}.$$

By $|\cdot|_{k,L_p(\Omega)}$, we shall denote the homogeneous kth order L^p-Sobolev semi-norm, i.e.,

$$|f|_{k,L_p(\Omega)} := \left(\sum_{|\alpha|_1 = k} \|D^\alpha f\|_{L_p(\Omega)}^p \right)^{1/p}.$$

Furthermore, it is clear that, due to edge effect, we lose some accuracy near the boundary. This corroborates numerical experiments performed by Beatson and Powell (see [P2]). Hence, the other goal of this paper is to show that when additional smoothness requirements and boundary conditions are met, the approximation order of the interpolation method can be improved at least twice. We now present two theorems which may be regarded as prototypes of the main results of this paper:

Theorem 1.1. *Let $s_{f,X}$ in* (1.1) *be the interpolant to f using the "shifted" surface spline ϕ_λ. Assume that the parameter λ in the basis function ϕ_λ is chosen to be proportional to h. Then, there is a positive constant c, independent of X, such that for every $f \in W_2^m(\Omega) \cap W_\infty^m(\Omega)$, we have an error bound of the form*

$$\|f - s_{f,X}\|_{L_p(\Omega)} \leq ch^{\gamma_p} |f|_{m,L_2(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty,$$

with

$$\gamma_p := \min(m, m - d/2 + d/p). \tag{1.8}$$

Furthermore, if $f \in W_2^k(\Omega) \cap W_\infty^k(\Omega)$ *with* $\max(0, d/2 - d/p) < k < m$, *then*

$$\|f - s_{f,X}\|_{L_p(\Omega)} = o(h^{\gamma_p - m + k}).$$

Theorem 1.2. *Suppose that* $D^\alpha f$ *is supported in* Ω *and bounded for any* $\alpha \in \mathbb{Z}_+^d$ *with* $|\alpha|_1 = 2m$. *Then, with the same notation and conditions as in Theorem 1.1, we have an improved error estimate*

$$\|f - s_{f,X}\|_{L_p(\Omega)} \le c_f h^{m+\gamma_p},$$

with a constant c_f *depending on* f.

The outline of the paper is as follows: In section 2, we discuss the properties of the Fourier transform $\hat{\phi}_\lambda$ and then discuss the extension of a function $f \in W_p^k(\Omega)$ to a function in the space $W_p^k(\mathbb{R}^d)$. In section 3, we provide some basic estimates for the error $f - s_{f,X}$; these will be used in the following sections to quantify the approximation power of $s_{f,X}$. Section 4 is devoted to establishing the asymptotic decay of the error between f and $s_{f,X}$ in the Sobolev space $W_p^k(\Omega)$ in the sense of L_p-norm. In section 5, we will observe that, under some suitable boundary and smoothness conditions of f, the accuracy of the interpolation method can be at least doubled.

2. Preliminaries

The basis function ϕ_λ in (1.3) grows polynomially for large argument. Then, in the sense of tempered distributions, ϕ_λ has the following generalized Fourier transform (see [GS]):

$$\hat{\phi}_\lambda(\theta) = c_{m,d}|\theta|^{-2m}\tilde{K}_m(|\lambda\theta|), \tag{2.1}$$

where $c_{m,d}$ is a positive constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|)$ with $K_\nu(|t|)$ the modified Bessel function of order ν. From [AS], we find that

$$\begin{aligned} &\tilde{K}_\nu(|t|) \in C^{2\nu-1}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \{0\}),\\ &\tilde{K}_\nu(|t|) > 0, \quad \text{and} \quad \tilde{K}_\nu(|t|) \approx e^{-|t|}(1 + |t|^{(\nu-1)/2}). \end{aligned} \tag{2.2}$$

We will see in the following sections that the Fourier transform $\hat{\phi}_\lambda$ is an important ingredient in our error estimates between f and $s_{f,X}$. Furthermore, for the given basis function ϕ_λ, there arises a function space

$$\mathcal{F}_\phi := \big\{g : |g|_{\phi_\lambda}^2 := \int_{\mathbb{R}^d} \frac{|\hat{g}(\theta)|^2}{\hat{\phi}_\lambda(\theta)} d\theta < \infty\big\}, \tag{2.3}$$

which is called the "native" space for ϕ_λ. This function space $\mathcal{F}_\phi$ is equipped with the semi-inner product

$$(f, g)_{\phi_\lambda} := \int_{\mathbb{R}^d} \hat{f}(\theta)\bar{\hat{g}}(\theta)\hat{\phi}_\lambda^{-1} d\theta.$$

The kernel of the semi-inner product is $\Pi_m(\mathbb{R}^d)$. It is known that the interpolant $s_{f,X}$ in (1.1) is the best approximation to $f \in \mathcal{F}_\phi$ from the space $\text{span}\{\phi_\lambda(\cdot - x_j) : x_j \in X\} + \Pi_m$ with respect to the $|\cdot|_{\phi_\lambda}$-semi norm. Thus, we have the property

$$|f - s_{f,X}|_{\phi_\lambda}^2 + |s_{f,X}|_{\phi_\lambda}^2 = |f|_{\phi_\lambda}^2, \tag{2.4}$$

which gives the relations

$$|s_{f,X}|_{\phi_\lambda} \le |f|_{\phi_\lambda} \quad \text{and} \quad (f - s_{f,X}, s_{f,X})_{\phi_\lambda} = 0. \tag{2.5}$$

For a given function $f \in W_p^k(\Omega)$, the assumptions on Ω in section 1 assure the existence of a function on $\mathbb{R}^d$ whose restriction to Ω agrees with f. The following results are cited from the literature.

Theorem 2.1 (Brenner and Scott, [BrS]). *Suppose that Ω has a Lipschitz boundary. Then for every function $f \in W_p^k(\Omega)$, there is an extension mapping $E : W_p^k(\Omega) \to W_p^k(\mathbb{R}^d)$, defined for all nonnegative integers k and real numbers p in the range $1 \le p \le \infty$, satisfying $Ef|_\Omega = f$ for all $f \in W_p^k(\Omega)$ and*

$$\|f^\Omega\|_{k,L_p(\mathbb{R}^d)} \le c\,\|f\|_{k,L_p(\Omega)},$$

where the constant c is independent of f.

The extension map E depends on Ω. Hence, in the following, we will use the notation $f^\Omega := E_\Omega(f) = E(f)$.

Lemma 2.2 (Light and Wayne, [LW]). *Let B be any ball of radius r in $\mathbb{R}^d$. Let $f \in W_2^k(B)$. Then there exists a unique function $f^B = E_B(f)$ such that $|f^B|_{k,L_2(\mathbb{R}^d)} < \infty$ and $f^B|_B = f$. Moreover, there exists a constant c, independent of B, such that for all $f \in W_2^k(B)$,*

$$|f^B|_{k,L_2(\mathbb{R}^d)} \le c\,|f|_{k,L_2(B)}.$$

The construction of a suitable extension of $f \in W_p^k(\Omega)$ to a function on $\mathbb{R}^d$ can be done in two steps. First, according to Theorem 2.1, there exists a function $f^\Omega \in W_p^k(\mathbb{R}^d)$ such that $f^\Omega|_\Omega = f$. Second, we let σ_Ω be a C^k-cutoff function such that $\sigma_\Omega(x) = 1$ for $x \in \Omega$ and $\sigma_\Omega(x) = 0$ for $|x| > r$ with a sufficiently large $r > 0$. Then, we define an extension f^o by

$$f^o := \sigma_\Omega f^\Omega.$$

Of course, f^o is compactly supported and $f^o(x) = f(x)$ for $x \in \Omega$. Indeed, for a large part of this paper, we wish to work with f^o and not f. For convenience, we will henceforth write f for f^o. Therefore, in this paper, we assume, *without great loss*, that an approximand $f \in W_p^k(\Omega)$ is supported on a sufficiently large compact set in $\mathbb{R}^d$, and that $f \in W_p^k(\mathbb{R}^d)$.

3. Basic results

In this section, we provide some basic estimates for the error $f - s_{f,X}$ in the Sobolev space $W_2^k(\Omega)$. These results will be used in the following sections to estimate the approximation power of $s_{f,X}$. Before we advance our discussion further, we introduce the following definition.

Definition 3.1. A vector $(u_1(x), \ldots, u_N(x))$, $x \in \Omega$, is said to be ℓ_p-admissible on Ω if the following conditions hold:

(a) There exists a constant $c_1 > 0$ such that for any $x \in \Omega$, $u_j(x) = 0$ whenever $|x - x_j| > c_1 h$, with h the density of X as in (1.4).

(b) The set $\{\mathbf{u}(x) := (u_1(x), \ldots, u_N(x)) : x \in \Omega\}$ is bounded in $\ell_p(X)$, namely, there exists a constant $c_2 > 0$ such that for any $x \in \Omega$, $\|\mathbf{u}(x)\|_p \le c_2$ for any $x \in \Omega$.

If, in addition to (a) and (b), the vector $(u_1(x), \dots, u_N(x))$ also satisfies the polynomial reproduction property

$$\sum_{j=1}^{N} u_j(x)\, p(x_j) = p(x), \quad x \in \Omega, \quad p \in \Pi_n,$$

then we say that the vector is "ℓ_p-admissible for Π_n" on Ω.

Remark. In Definition 3.1, the centers in the set

$$X_x := \{x_j \in X \,:\, u_j(x) \neq 0\}, \quad x \in \Omega,$$

are assumed to be some "close neighbors" of x. Of course, the set X_x is required to have the nondegeneracy property for Π_n. This implies that $\#X_x$ should be no smaller than $\dim \Pi_n(\mathbb{R}^d)$, where $\#S$ denotes the number of elements of a set S. For examples of such ℓ_p-admissible vectors $(u_1(x), \dots, u_N(x))$, the reader is referred to the papers [L] and [WS].

Remark. For any $j = 1, \dots, N$, we find easily that the function u_j is supported in Ω and $\mathrm{vol}(\mathrm{supp}\, u_j) \le ch^d$ for some $c > 0$. In this section, we use the notation S_j for the smallest ball centered at x_j including $\mathrm{supp}\ u_j$. Then we have the following result:

Lemma 3.2. *Suppose $X = \{x_1, \dots, x_N\} \subset \Omega$, and let $(u_1(x), \dots, u_N(x))$ be ℓ_p-admissible for Π_n on Ω. Let S_j, $j = 1, \dots, N$, be as above. The following hold:*

(a) $\sum_{j=1}^{N} \chi_{S_j} \le c_n$, *with the constant c_n depending on n, where χ_S is the characteristic function of a set S.*
(b) $\Omega \subset \bigcup_{j=1}^{N} S_j$.
(c) $\mathrm{vol}(S_j) \le ch^d$ *for some $c > 0$.*

Proof. According to Definition 3.1 (a), for any $x \in \Omega$, there exists a constant $c > 0$ such that if $|x_j - x| > ch$, then $u_j(x) = 0$ and $\chi_{S_j}(x) = 0$. Then, from (1.6), we find that there exist at most M terms (M depends on n), say $u_{i_1}(x), \dots, u_{i_M}(x)$, such that $\chi_{S_{i_\ell}}(x) \neq 0$ with $\ell = 1, \dots, M$, which proves (a). The relations (b) and (c) are direct consequences of Definition 3.1. $\square$

Lemma 3.3. *Let $s_{f,X}$ in* (1.1) *be an interpolant to f using the basis function ϕ_λ. Let S_j with $j = 1, \dots, N$ be the smallest ball centered at x_j including* $\mathrm{supp}\, u_j$. *Then, for every $f \in W_2^m(S_j)$,*

$$\|f - s_{f,X}\|_{L_p(S_j)} \le ch^{m-d/2+d/p} |(f - s_{f,X})^{S_j}|_{m,L_2(\mathbb{R}^d)}.$$

Proof. Since $f - s_{f,X} \in W_2^m(S_j)$ for any $j = 1, \dots, N$, Theorem 2.1 shows that there exists an extension $(f - s_{f,X})^{S_j} \in W_2^m(\mathbb{R}^d)$ such that $|(f - s_{f,X})^{S_j}|_{m,L_2(\mathbb{R}^d)}$ is bounded and

$$(f - s_{f,X})|_{S_j} = (f - s_{f,X})^{S_j}|_{S_j}.$$

Now, let Tf be an interpolant to f on a set $X \cap S_j$ by using the surface spline ϕ_0; for simplicity, we use the notation Tf temporarily in this proof. Then, it is clear that for any $y \in X \cap S_j$ we have $(f - s_{f,X})^{S_j}(y) = 0$, implying that the interpolant $T(f - s_{f,X})^{S_j}$ is identically zero. Therefore,

$$(f - s_{f,X})^{S_j} = (f - s_{f,X})^{S_j} - T(f - s_{f,X})^{S_j}.$$

Note that the density of the set $X \cap S_j$ in S_j is bounded by a constant times h. Then it is known (e.g., see [WS] or [Y2]) that

$$\|(f - s_{f,X})^{S_j} - T(f - s_{f,X})^{S_j}\|_{L_\infty(S_j)} \le ch^{m-d/2}|(f - s_{f,X})^{S_j}|_{m,L_2(\mathbb{R}^d)}.$$

Using condition (c) of Lemma 3.2, we obtain

$$\begin{aligned}\|f - s_{f,X}\|_{L_p(S_j)} &\le \operatorname{vol}(S_j)^{1/p}\|(f - s_{f,X})^{S_j} - T(f - s_{f,X})^{S_j}\|_{L_\infty(S_j)} \\ &\le ch^{m-d/2+d/p}|(f - s_{f,X})^{S_j}|_{m,L_2(\mathbb{R}^d)},\end{aligned}$$

which completes the proof of the lemma. □

Theorem 3.4. *Let $s_{f,X}$ in (1.1) be an interpolant to f on X using the basis function ϕ_λ. Assume that γ_p is defined by (1.8) for $1 \le p \le \infty$. Then, there exists a constant $c > 0$, independent of X, such that for every function $f \in W_2^m(\Omega)$, we obtain an estimate of the form*

$$\|f - s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f - s_{f,X}|_{m,L_2(\mathbb{R}^d)}. \tag{3.1}$$

Proof. First, for $1 \le p \le \infty$, we claim the following inequality:

$$\|f - s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}\Big(\sum_{j=1}^N |(f - s_{f,X})^{S_j}|^2_{m,L_2(\mathbb{R}^d)}\Big)^{1/2}. \tag{3.2}$$

When $p = \infty$, this inequality is directly implied by Lemma 3.3. Hence, we assume that $1 \le p < \infty$. Let q be the exponent conjugate to p, i.e., $1/p + 1/q = 1$, and let $\mathbf{u}(x) := (u_1(x), \ldots, u_N(x))$, $x \in \Omega$, be an ℓ_q-admissible vector for Π_1 on Ω. Recalling that S_j is the smallest ball including supp u_j, we apply the property $\sum_{j=1}^N u_j(x) = 1$, $x \in \Omega$, to obtain the estimate

$$\begin{aligned}|f(x) - s_{f,X}(x)| &= |\sum_{j=1}^N u_j(x)(f - s_{f,X})(x)| \\ &\le \|\mathbf{u}(x)\|_q \left(\sum_{j=1}^N |\chi_{S_j}(x)(f - s_{f,X})(x)|^p\right)^{1/p}.\end{aligned}$$

By condition (b) of Definition 3.1, the term $\|\mathbf{u}(x)\|_q$ is uniformly bounded. Consequently,

$$\begin{aligned}\|f - s_{f,X}\|^p_{L_p(\Omega)} &\le c\sum_{j=1}^N \int_{S_j} |(f - s_{f,X})(x)|^p dx \\ &\le c\sum_{j=1}^N \operatorname{vol}(S_j)\,\|f - s_{f,X}\|^p_{L_\infty(S_j)} \\ &\le c'h^{p(m-d/2)+d}\sum_{j=1}^N |(f - s_{f,X})^{S_j}|^p_{m,L_2(\mathbb{R}^d)},\end{aligned}$$

where the last inequality is implied by the fact $\operatorname{vol}(S_j) \le ch^d$ (see Lemma 3.2) and a direct application of Lemma 3.3 with $p = \infty$. Now, since $N \le ch^{-d}$, we prove (3.2) by using the inequality $\big(\sum_{j=1}^N a_j^p\big)^{1/p} \le ch^{\min(0,d/2-d/p)}\big(\sum_{j=1}^N a_j^2\big)^{1/2}$ (see [H]). Consequently, using Lemmas 2.2 and 3.2, we have

$$\begin{aligned}\sum_{j=1}^N |(f-s_{f,X})^{S_j}|^2_{m,L_2(\mathbb{R}^d)} &\le c\sum_{|\alpha|_1=m}\sum_{j=1}^N\int_{S_j}|D^\alpha(f-s_{f,X})(x)|^2dx\\ &= c\sum_{|\alpha|_1=m}\int_{\mathbb{R}^d}\sum_{j=1}^N|\chi_{S_j}(x)D^\alpha(f-s_{f,X})(x)|^2dx\\ &\le c'\sum_{|\alpha|_1=m}\int_{\mathbb{R}^d}|D^\alpha(f-s_{f,X})(x)|^2dx.\end{aligned}$$

This together with (3.2) gives the desired estimate:

$$\|f-s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f-s_{f,X}|_{m,L_2(\mathbb{R}^d)}.$$ □

Corollary 3.5. *Assume that f is a function in the space $\mathcal{F}_\phi$. Then, under the same conditions and notation of Theorem 3.4, we have*

$$\|f-s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f-s_{f,X}|_{\phi_\lambda} \le ch^{\gamma_p}|f|_{\phi_\lambda},$$

where γ_p is defined by (1.8).

Proof. Remembering the properties of the Fourier transform $\hat{\phi}_\lambda$ in (2.1) and (2.2), we find that

$$\hat{\phi}_\lambda^{-1} = c_{m,d}|\theta|^{2m}\tilde{K}_m^{-1}(|\lambda\theta|) \ge c|\theta|^{2m} \tag{3.3}$$

for some $c>0$, where $\tilde{K}_m(|t|) := |t|^m K_m(|t|)$ with $K_m(|t|)$ the modified Bessel function of order m. Then the inequality (3.3) and Theorem 3.4 imply the bound

$$\|f-s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f-s_{f,X}|_{m,L_2(\Omega)} \le ch^{\gamma_p}|f-s_{f,X}|_{\phi_\lambda}.$$

The required result follows from the inequality $|f-s_{f,X}|_{\phi_\lambda} \le |f|_{\phi_\lambda}$, which is an easy consequence of (2.4). □

When $\lambda=0$ in ϕ_λ, the basis function ϕ_0 becomes the surface spline $\phi_0 := (-1)^{\lceil m-d/2\rceil}|\cdot|^{2m-d}$ if d is odd, and $\phi_0 := (-1)^{m-d/2+1}|\cdot|^{2m-d}\log|\cdot|$ if d is even, where $m>d/2$ in both cases. Then we have the following estimate, which is equivalent to Duchon's results [Du, page 334].

Corollary 3.6. *Let $s_{f,X}$ in* (1.1) *be an interpolant to f on X using the surface spline function ϕ_0. Then, for every $f\in W_2^m(\Omega)$, there is an error bound of the form*

$$\|f-s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f|_{m,L_2(\mathbb{R}^d)},$$

where γ_p is given by (1.8).

Proof. The generalized Fourier transform of ϕ_0 is of the form $\hat{\phi}_0 = c_{m,d}|\cdot|^{-2m}$ (see (2.1)). Then it follows that $|f|_{\phi_0} \le c|f|_{m,L_2(\mathbb{R}^d)}$. Hence, this corollary is immediate from Corollary 3.5. □

4. L_p-error estimates in Sobolev space

We shall now prove the approximation order of interpolation to functions f in a Sobolev space. To this end, for a given function f, we first approximate f by a band-limited function

$$f_H := f_{H,h} := \sigma(h\cdot)^\vee * f \in \mathcal{F}_\phi, \tag{4.1}$$

where $\sigma : \mathbb{R}^d \to [0,1]$ is a nonnegative C^∞ cutoff function whose support σ lies in the Euclidean ball B_1 with $\sigma = 1$ on $B_{1/2}$ and $\|\sigma\|_{L_\infty(\mathbb{R}^d)} = 1$. Then, it is useful for error analysis to divide $f - s_{f,X}$ into two parts,

$$f - s_{f,X} = (f_H - s_{f_H,X}) + (f_T - s_{f_T,X}), \tag{4.2}$$

where

$$f_T := f_{T,h} := f - \sigma(h\cdot)^\vee * f. \tag{4.3}$$

Since the function f_H belongs to the space $\mathcal{F}_\phi$, Corollary 3.5 can be applied directly to estimate the error $f_H - s_{f_H,X}$. As seen from (4.1), the function f_H depends on h. In the following lemma, we estimate the magnitude of $|f_H|_{\phi_\lambda}$ for small $h > 0$.

Lemma 4.1. *Suppose that the parameter λ in ϕ_λ satisfies the relation $\lambda = \rho h$ for a fixed $\rho \geq 0$. Then, for every $f \in W_2^m(\Omega)$,*

$$|f_H|_{\phi_\lambda} \leq c_\rho |f|_{m,L_2(\mathbb{R}^d)},$$

where the constant c_ρ is independent of h but depends on ρ. Furthermore, if $f \in W_2^k(\Omega)$ with $\max(0, d/2 - d/p) < k < m$, *then*

$$|f_H|_{\phi_\lambda} = o(h^{k-m}), \quad h \to 0.$$

Proof. First, let us assume that $f \in W_2^m(\Omega)$. From the definition of f_H in (4.1), it is clear that $\hat{f}_H = \sigma(h\cdot)\hat{f}$. From (2.1) and (2.3), we find, via Parseval's identity, that

$$\begin{aligned} |f_H|^2_{\phi_\lambda} &= c_{m,d} \int_{\mathbb{R}^d} \frac{|\theta|^{2m}}{\tilde{K}_m(|\lambda\theta|)} \sigma^2(h\theta)|\hat{f}(\theta)|^2 d\theta \\ &\leq c \left\| \frac{\sigma^2(\cdot)}{\tilde{K}_m(|\rho\cdot|)} \right\|_{L_\infty(\mathbb{R}^d)} |f|^2_{m,L_2(\mathbb{R}^d)} \\ &\leq c' \, |f|^2_{m,L_2(\mathbb{R}^d)}, \end{aligned}$$

where the first inequality is valid by the condition $\lambda = \rho h$ with $\rho > 0$.

Next, let us consider the case $f \in W_2^k(\Omega)$ with $\max(0, d/2 - d/p) < k < m$. Denote $|\theta|^{2m} =: \sum_{|\nu|_1 = m} c_\nu \theta^{2\nu}$ for some suitable constants $c_\nu > 0$. For each $\nu \in \mathbb{Z}_+^d$ with $|\nu|_1 = m$, we can write $\nu = \alpha + \beta$ with $\alpha, \beta \in \mathbb{Z}_+^d$ such that $|\alpha|_1 = m - k > 0$ and $|\beta|_1 = k$. Here, for simplicity, we use the following abbreviation:

$$\mathcal{K}_{h,\alpha} := \frac{(\cdot)^{2\alpha}\sigma^2(h\cdot)}{\tilde{K}_m(|\lambda\cdot|)}.$$

Then, rewriting $c_\nu =: c_{m,\alpha,\beta}$, we get

$$\begin{aligned}|f_H|^2_{\phi_\lambda} &= i^{-2|\beta|_1} \sum_{|\alpha+\beta|_1=m} c_{m,\alpha,\beta} \int_{\mathbb{R}^d} \mathcal{K}_{h,\alpha}(\theta) |\widehat{D^\beta f}(\theta)|^2 d\theta \\ &\leq c \sum_{|\alpha+\beta|_1=m} \int_{\mathbb{R}^d} \left| D^\beta f(-\theta)(\mathcal{K}^\vee_{h,\alpha} * D^\beta f)(\theta) \right| d\theta \\ &\leq c|f|_{k,L_2(\mathbb{R}^d)} \sum_{|\alpha+\beta|_1=m} \|\mathcal{K}^\vee_{h,\alpha} * D^\beta f\|_{L_2(\mathbb{R}^d)},\end{aligned}$$

where the first inequality is valid by Parseval's formula and the relation $(gh)^\vee = g^\vee * h^\vee$ for suitable g and h, and the second follows from Hölder's inequality. Also, using the property $\int_{\mathbb{R}^d} g(t)dt = \hat{g}(0)$ for any $g \in L_1(\mathbb{R}^d)$, we note that

$$\int_{\mathbb{R}^d} \mathcal{K}^\vee_{h,\alpha}(\theta) d\theta = \mathcal{K}_{h,\alpha}(0) = 0, \quad |\alpha|_1 = m-k > 0,$$

whence we get the equation

$$\begin{aligned}\mathcal{K}^\vee_{h,\alpha} * D^\beta f(t) &= \int_{\mathbb{R}^d} \left(\frac{(\cdot)^{2\alpha}\sigma^2(h\cdot)}{\tilde{K}_m(|\lambda\cdot|)} \right)^\vee (\theta) \big(D^\beta f(t-\theta) - D^\beta f(t) \big) d\theta \\ &= h^{-2(m-k)} \int_{\mathbb{R}^d} \left(\frac{(\cdot)^{2\alpha}\sigma^2(\cdot)}{\tilde{K}_m(|\rho\cdot|)} \right)^\vee (\theta) (D^\beta f(t-h\theta) - D^\beta f(t)) d\theta\end{aligned}$$

by a change of variables and the relation $\lambda = \rho h$. Consequently, by using the generalized Minkowski's inequality, we have an estimate for $|f_H|_{\phi_\lambda}$ as follows:

$$|f_H|^2_{\phi_\lambda} \leq ch^{-2(m-k)} \int_{\mathbb{R}^d} \left| \left(\frac{(\cdot)^{2\alpha}\sigma^2(\cdot)}{\tilde{K}_m(|\rho\cdot|)} \right)^\vee (\theta) \right| \|D^\beta f(\cdot - h\theta) - D^\beta f\|_{L_2(\mathbb{R}^d)} d\theta.$$

It is known (see [F, Proposition 8.5]) that $\|D^\beta f(\cdot - h\theta) - D^\beta f\|_{L_2(\mathbb{R}^d)} \to 0$ as $h \to 0$. Therefore, by applying the Lebesgue dominated convergence theorem, we have the convergence property $|f_H|_{\phi_\lambda} = o(h^{k-m})$ as $h \to 0$. □

The following lemma treats the error $f_H - s_{f_H,X}$.

Lemma 4.2. *Assume that the parameter λ in ϕ_λ satisfies the relation $\lambda = \rho h$ for a fixed $\rho \geq 0$. Let γ_p be defined as in (1.8) for $1 \leq p \leq \infty$. Then, for every $f \in W_2^m(\Omega)$,*

$$\|f_H - s_{f_H,X}\|_{L_p(\Omega)} \leq ch^{\gamma_p} |f|_{m,L_2(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty.$$

Furthermore, if $f \in W_2^k(\Omega)$ with $\max(0, d/2 - d/p) < k < m$, *then*

$$\|f_H - s_{f_H,X}\|_{L_p(\Omega)} = o(h^{\gamma_p - m + k}).$$

Proof. This is a direct consequence of Corollary 3.5 and Lemma 4.1. □

Next we shall estimate the error $f_T - s_{f_T,X}$. Of course, there is no guarantee that the function f_T belongs to the space $\mathcal{F}_\phi$, and this makes the analysis more complicated than the previous case. In order to obtain the required error estimate, we employ the interpolant, say $g_{f,X}$, using the (scaled) Gaussian function

$$\varphi_q(x) := \varphi(x/q) := e^{-|x|^2/q^2}, \tag{4.4}$$

where q is the separation distance (1.5) within X (see, for example, [NW, page 80]). In this case $m = 0$ (see (1.2)), and hence the interpolant $g_{f,X}$ is of the form

$$g_{f,X}(x) = \sum_{j=1}^{N} \beta_j \varphi_q(x - x_j). \tag{4.5}$$

Also, it is well known that the Gaussian function φ_q is a strictly conditionally positive definite function; that is, the matrix $\mathbf{A_g} = (\phi_q(x_i - x_j))_{i,j=1,\dots,N}$ is nonsingular for every choice of distinct points $x_1, \dots, x_N$ in $\mathbb{R}^d$ (e.g., see [P1]). Indeed, the separation distance q is employed to use the stability results on Gaussian interpolation. The following lemma depends on the estimate of the inverse matrix $\mathbf{A_g}^{-1}$, though it is not explained explicitly. The reader is referred to the papers [BSW] and [Y2] for more details.

Lemma 4.3. *Let q be the separation distance within X, and φ_q the dilated Gaussian function defined above. Let $g_{f,X}$ in (4.5) be the interpolant to f on X. Then, for every $f \in L_\infty(\Omega)$, we have the following properties:*

(a) $\|g_{f,X}\|_{L_p(\Omega)} \le c\|f\|_{L_\infty(\Omega)}$*, where $1 \le p \le \infty$ and the constant c is independent of X.*
(b) *If the parameter λ in ϕ_λ satisfies the relation $\lambda = \rho h$ for a fixed $\rho > 0$, then* $|g_{f,X}|_{\phi_\lambda} \le ch^{-m}\|f\|_{L_\infty(\Omega)}$*, with c independent of X.*

Proof. Let

$$M_q := \Big\| \sum_{j=1}^{N} \varphi_q(\cdot - x_j) \Big\|_{L_\infty(\mathbb{R}^d)}. \tag{4.6}$$

The finiteness of M is a consequence of the decay of the Gaussian function φ_q and the fact that X is a q-separated set. Then, denoting $\mathbf{b_f} := (\beta_1, \dots, \beta_N)$, we derive from the definition of $g_{f,X}$ in (4.5) that

$$\begin{aligned}
\|g_{f,X}\|^p_{L_p(\Omega)} &= \int_\Omega \Big| \sum_{j=1}^{N} \beta_j \varphi_q(x - x_j) \Big|^p dx \\
&\le M_q^{p-1} \|\mathbf{b_f}\|^p_\infty \sum_{j=1}^{N} \int_\Omega \varphi_q(x - x_j) dx \\
&\le M_q^{p-1} \|\mathbf{b_f}\|^p_\infty N q^d \|\varphi\|_{L_1(\mathbb{R}^d)} \\
&\le c M_q^{p-1} \|\mathbf{b_f}\|^p_\infty,
\end{aligned}$$

the last inequality being valid by the conditions $N \le ch^{-d}$ and $q \le h$. Here, the constant c is independent of X. Furthermore, with the help of Lemma 2.5 in [Y2], we get $\|\mathbf{b_f}\|_\infty \le c\|f\|_{L_\infty(\Omega)}$. This completes the proof of (a).

Now we prove (b). Recalling the explicit form of $|\cdot|_{\phi_\rho}$ in (2.3), we deduce by a change of variables that

$$|g_{f,X}(h\cdot)|^2_{\phi_\rho} = |g_{f,X}|^2_{\phi_\rho(\cdot/h)} \le c \int_{\mathbb{R}^d} \Big| \sum_{j=1}^{N} \beta_j e^{ix_j\cdot\theta} \Big|^2 \hat{\varphi}_q(\theta) d\theta$$

for some $c > 0$, where the inequality is implied by the condition that $\hat{\varphi}_q/\hat{\phi}_\rho(\cdot/h)$ is uniformly bounded. Now we claim that

$$\hat{\varphi}^2_{q/\sqrt{2}} = q^d(\pi/4)^{d/2}\hat{\varphi}_q.$$

In fact, remembering the Fourier transform $\hat{\varphi}(\theta) = \pi^{d/2}e^{-|\theta|^2/4}$ with φ in (4.4), this is proved by the following direct calculations:

$$\hat{\varphi}^2_{q/\sqrt{2}}(\theta) = (q/\sqrt{2})^{2d}\hat{\varphi}^2(q\theta/\sqrt{2}) = (\pi/2)^d e^{-q^2|\theta|^2/4}q^{2d} = q^d(\pi/4)^{d/2}\hat{\varphi}_q(\theta).$$

Hence, we use this claim to derive the relation

$$\begin{aligned} |g_{f,X}(h\cdot)|^2_{\phi_\rho} &\le cq^{-d}\int_{\mathbb{R}^d}\Big|\sum_{j=1}^N \beta_j e^{ix_j\cdot\theta}\hat{\varphi}_{q/\sqrt{2}}(\theta)\Big|^2 d\theta \\ &= cq^{-d}\int_{\mathbb{R}^d}\Big|\sum_{j=1}^N \beta_j \varphi_{q/\sqrt{2}}(x-x_j)\Big|^2 dx. \end{aligned}$$

Thus,

$$\begin{aligned} |g_{f,X}(h\cdot)|^2_{\phi_\rho} &\le cM_{q/\sqrt{2}}\|\mathbf{b_f}\|^2_\infty q^{-d}\int_{\mathbb{R}^d}\Big|\sum_{j=1}^N \varphi_{q/\sqrt{2}}(x-x_j)\Big|dx \\ &\le cM_{q/\sqrt{2}}N\|\mathbf{b_f}\|^2_\infty q^{-d}\|\varphi_{q/\sqrt{2}}\|_{L_1(\mathbb{R}^d)} \\ &\le c'h^{-d}\|f\|^2_{L_\infty(\Omega)} \end{aligned}$$

with $M_{q/\sqrt{2}}$ given by (4.6), where the last inequality is true by the condition $N \le ch^{-d}$. Then, using the relation $\lambda = \rho h$, we derive by a change of variables that

$$|g_{f,X}|_{\phi_\lambda} = h^{-m+d/2}|g_{f,X}(h\cdot)|_{\phi_\rho} \le ch^{-m}\|f\|_{L_\infty(\Omega)},$$

as desired. □

Lemma 4.4. *Assume that $f \in W^k_\infty(\Omega)$ with $k > 0$, and define $\tilde{f}$ as follows:*

$$\tilde{f} := \tilde{f}_h := h^{-k}f_T,$$

with f_T given by (4.3). Then $\|\tilde{f}\|_{L_p(\mathbb{R}^d)} = o(1)$ as h tends to 0.

Remark. We remind the reader of the assumption that every function f in the space $W^m_p(\Omega)$ is supported in a sufficiently large compact domain in $\mathbb{R}^d$ (see Section 2). Hence, every function $f \in W^m_\infty(\Omega)$ belongs to the space $W^m_p(\Omega)$ for any p in $[1, \infty]$.

Proof. This is a restatement of the fact, proved in [Y1], that $\|f_T\|_{L_p(\mathbb{R}^d)} = o(h^k)$ as $h \to 0$. □

Now we are ready to estimate the error $f_T - s_{f_T,X}$.

Lemma 4.5. *Let $s_{f,X}$ in (1.1) be an interpolant to f on X using the basis function ϕ_λ. Suppose that the parameter λ in ϕ_λ is chosen to be proportional to the density of X, i.e., $\lambda = \rho h$, $\rho > 0$. Then, for every $f \in W^k_\infty(\Omega)$ with $\max(0, d/2 - d/p) < k \le m$, we have an error bound of the form*

$$\|f_T - s_{f_T,X}\|_{L_p(\Omega)} = o(h^{\gamma_p - m + k}),$$

where γ_p is defined by (1.8).

Proof. Using the function $\tilde{f} = h^{-k} f_T$ in Lemma 4.4, we can estimate the error $f_T - s_{f_T,X}$ as follows:

$$h^{-k}\|f_T - s_{f_T,X}\|_{L_p(\Omega)} \le \|\tilde{f}\|_{L_p(\Omega)} + \|g_{\tilde{f},X}\|_{L_p(\Omega)} + \|g_{\tilde{f},X} - s_{\tilde{f},X}\|_{L_p(\Omega)}. \tag{4.7}$$

According to Lemma 4.3 and Lemma 4.4, the terms $\|\tilde{f}\|_{L_p(\Omega)}$ and $\|g_{\tilde{f},X}\|_{L_p(\Omega)}$ converge to 0 as h tends to 0. So it remains to show that the last term on the right-hand side of (4.7) is $o(h^{\gamma_p - m})$. To this end, we first note that, since $\tilde{f}(x_j) = g_{\tilde{f},X}(x_j)$ for every $1 \le j \le N$, the interpolant $s_{\tilde{f},X}$ can be considered as an interpolant to the function $g_{\tilde{f},X}$; that is, $s_{\tilde{f},X} = s_{g_{\tilde{f},X},X}$. Therefore, it follows from Corollary 3.5 that

$$\|s_{\tilde{f},X} - g_{\tilde{f},X}\|_{L_p(\Omega)} = \|s_{g_{\tilde{f},X},X} - g_{\tilde{f},X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|g_{\tilde{f},X}|_{\phi_\lambda} = o(h^{\gamma_p - m}),$$

the final bound resulting from Lemmas 4.3 and 4.4. □

The next theorem is the main result of this section.

Theorem 4.6. *Let $s_{f,X}$ in* (1.1) *be an interpolant to f on X using the basis function ϕ_λ. Let $\gamma_p = \min(m, m - d/2 + p/d)$ with $1 \le p \le \infty$. Assume that:*

(a) *There exists a constant $\eta > 0$, independent of X, such that $h/q \le \eta$, where q is the separation distance of X (see* (1.5)*), and h is the density of X (see* (1.4)*).*

(b) *The parameter λ is chosen to satisfy the relation $\lambda = \rho h$ for a fixed $\rho > 0$.*

Then, there is a constant $c > 0$, independent of X, such that for every $f \in W_2^m(\Omega) \cap W_\infty^m(\Omega)$, we have an error bound of the form

$$\|f - s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f|_{m,L_2(\mathbb{R}^d)}.$$

Furthermore, for every $f \in W_2^k(\Omega) \cap W_\infty^k(\Omega)$ with $\max(0, d/2 - d/p) < k < m$, we have

$$\|f - s_{f,X}\|_{L_p(\Omega)} = o(h^{\gamma_p - m + k}).$$

Proof. This follows from (4.2), Lemma 4.2, and Lemma 4.5. □

Remark. Since the domain Ω is a bounded set in $\mathbb{R}^d$, the space $W_2^m(\Omega) \cap W_\infty^m(\Omega)$ coincides with $W_\infty^m(\Omega)$. However, in order to indicate that the error bounds depend on the norm $|f|_{m,2}$, we preferred to write $W_2^m(\Omega) \cap W_\infty^m(\Omega)$.

Remark. When $\lambda = 0$ in ϕ_λ, the basis function ϕ_0 becomes the surface spline. In this case, we find that the native space $\mathcal{F}_{\phi_0}$ is identically equal to the m-th order homogeneous Sobolev space. Hence, if $f \in W_2^m(\Omega)$, the error $f - s_{f,X}$ can be estimated directly (as in the proof of Lemma 4.2) without splitting f into two functions f_H and f_T.

Corollary 4.7. *Let $s_{f,X}$ in* (1.1) *be an interpolant to f on X using the surface spline function ϕ_0. Then, for every $f \in W_2^m(\Omega)$, there is an error bound of the form*

$$\|f - s_{f,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f|_{m,L_2(\mathbb{R}^d)},$$

where γ_p is defined by (1.8)*. Furthermore, for every $f \in W_2^k(\Omega) \cap W_\infty^k(\Omega)$ with $\max(0, d/2 - d/p) < k < m$,*

$$\|f - s_{f,X}\|_{L_p(\Omega)} = o(h^{\gamma_p - m + k}).$$

Note that the estimate in the first part of the above corollary is equivalent to known results (see [Du], [MN2] or [WS]).

5. Improved error bounds of interpolation method

When X is an infinite square grid of mesh size h and Ω is taken as all of $\mathbb{R}^d$, it is known that the "shifted" surface spline cardinal interpolation enjoys the convergence property $O(h^{2m})$ provided that f is sufficiently smooth (see [Bu2], [P1]). However, with a finite number of interpolation points, we obtained the order of accuracy $O(h^{\gamma_p})$ with $\gamma_p = \min(m, m - d/2 + d/p)$ in section 3. Usually, owing to the edge effects, one must lose some order of accuracy near the boundary. This corroborates experimental evidence reported by Powell and Beatson. Hence, the purpose of this section is to show that if additional smoothness requirements and boundary conditions are met, the approximation order of interpolation method can be at least doubled. For this purpose, we have to restrict ourselves to the functions f in the space

$$V_p^{2m}(\Omega) := \{f \in W_p^{2m}(\Omega) \,:\, \mathrm{supp} D^\alpha f \subset \Omega, \quad |\alpha|_1 = 2m\}.$$

Before we proceed further, let us introduce a bell-shaped function ψ_λ which is obtained by applying a suitable difference operator to ϕ_λ. The actual form of ψ_λ is as follows:

$$\psi_\lambda(x) := \sum_{\alpha \in \mathcal{G}} \mu(\alpha)\phi_\lambda(x - \alpha), \tag{5.1}$$

where $\mathcal{G}$ is a finite subset of a scaled uniform grid $\delta\mathbb{Z}^d$ with $\delta > 0$ and $(\mu(\alpha))_{\alpha\in\mathcal{G}}$ is a localization sequence. From now on, we assume that the localized function ψ_λ satisfies the following conditions:

$$\begin{aligned} &(a)\ \hat{\psi}_\lambda \in C^{4m-1}(\mathbb{R}^d), \quad D^\alpha \hat{\psi}_\lambda(0) = \delta_{\alpha,0}, \quad |\alpha|_1 < 2m; \\ &(b)\ \sup_x (1 + |x|)^{2m+d+k} |\psi_\lambda(x)| < \infty, \quad k > 0. \end{aligned} \tag{5.2}$$

In order to ensure the existence of ψ_λ (actually, the sequence $(\mu(\alpha))_{\alpha\in\mathcal{G}}$), the reader is referred to the article [DJLR].

Lemma 5.1. *Let ψ_λ be defined as above. Then, the map $f \mapsto \psi_\lambda * f$ reproduces Π_{2m}, i.e., $\psi_\lambda * p = p$ for any polynomial $p \in \Pi_{2m}$.*

Proof. It suffices to prove that $\int_{\mathbb{R}^d} \psi_\lambda(x - t)t^\nu dt = x^\nu$ for any $|\nu|_1 < 2m$. For this proof, using the property $\int_{\mathbb{R}^d} f(t)dt = \hat{f}(0)$ for any $f \in L_1(\mathbb{R}^d)$, we derive from (5.2)(b) that

$$\begin{aligned} \int_{\mathbb{R}^d} \psi_\lambda(x - t)t^\nu dt &= \big((\cdot)^\nu \psi_\lambda(\cdot - x)\big)^\wedge(0) \\ &= (-i)^{-\nu}(D^\nu(\hat{\psi}_\lambda e_{-ix}))(0), \quad e_{-ix} : \theta \mapsto e^{-ix\cdot\theta}, \end{aligned} \tag{5.3}$$

the second equality being valid because $\widehat{(\cdot)^\nu f} = (-i)^{-\lambda} D^\lambda \hat{f}$ for a sufficiently smooth $\hat{f}$. Then, letting $\nu = \alpha + \beta$ with $\alpha, \beta \in \mathbb{Z}_+^d$, we can write

$$\begin{aligned} (D^\nu(\hat{\psi}_\lambda e_{-ix}))(0) &= \sum_{\alpha+\beta=\nu} c_{\alpha,\beta}(D^\alpha \hat{\psi}_\lambda)(0)(D^\beta e_{-ix})(0) \\ &= \sum_{\alpha+\beta=\nu} c_{\alpha,\beta}(D^\alpha \hat{\psi}_\lambda)(0)(-ix)^\beta \end{aligned}$$

for some suitable coefficients $c_{\alpha,\beta}$. According to (5.2), $(D^\alpha \hat{\psi}_\lambda)(0) = 0$ for any $|\alpha|_1 > 0$. Moreover, since $c_{0,\nu} = 1$, we have

$$(D^\nu(\hat{\psi}_\lambda e_{-ix}))(0) = (-ix)^\nu.$$

Combining this identity with (5.3), we obtain the required result. $\square$

With these preliminaries in place, let us turn to the estimate of the error $f - s_{f,X}$. First, for any function $f \in V_p^{2m}(\Omega)$ with $1 \le p \le \infty$, we approximate f by

$$f^* := f^*_{\rho,h} := (\psi_\rho * f(h\cdot))(\cdot/h), \tag{5.4}$$

where $\rho = \lambda/h$, and then we split the error $f - s_{f,X}$ as follows:

$$\|f - s_{f,X}\|_{L_p(\mathbb{R}^d)} \le \|f - f^*\|_{L_p(\Omega)} + \|f^* - s_{f^*,X}\|_{L_p(\Omega)} + \|s_{f^*-f,X}\|_{L_p(\Omega)}.$$

It is essential that f^* should approximate f better as $h \to 0$. The following lemma illustrates the approximation behavior of f^* to f.

Lemma 5.2. *Let f^* be defined as in* (5.4). *Then there exists a constant c_ρ, independent of h, such that for every $f \in V_p^{2m}(\Omega)$ with $1 \le p \le \infty$, we have*

$$\|f - f^*\|_{L_p(\Omega)} \le c_\rho h^{2m} |f|_{2m,L_p(\Omega)}.$$

Proof. From (5.2), it is easy to check that $\int_{\mathbb{R}^d} \psi_\rho(\cdot/h - t)dt = \hat{\psi}_\rho(0) = 1$ for any $h > 0$. Hence, the definition $f^* = (\psi_\rho * f(h\cdot))(\cdot/h)$ leads to the identity

$$f^*(x) - f(x) = \int_{\mathbb{R}^d} \psi_\rho(x/h - t)(f(ht) - f(x))dt. \tag{5.5}$$

Then, taking the Taylor expansion of $f(\vartheta)$ about $\vartheta = x$, we get the expression

$$f(ht) - f(x) = \sum_{0<|\alpha|_1<2m} (ht - x)^\alpha D^\alpha f(x)/\alpha! + R_{2m} f(x,t) \tag{5.6}$$

with the remainder in the integral form

$$R_{2m} f(x,t) := \sum_{|\alpha|_1=2m} (ht - x)^\alpha \int_0^1 2m(1-y)^{2m-1}(D^\alpha f)((1-y)x + yht)dy/\alpha!.$$

Invoking the polynomial reproduction property $\psi_\rho * p = p$ for any $p \in \Pi_{2m}$ (see Lemma 5.1), it is immediate that $\int_{\mathbb{R}^d} \psi_\rho(x/h - t)(x/h - t)^\alpha dt = 0$ with $0 < |\alpha|_1 < 2m$. Hence, the integral of $\psi_\rho(x/h - t)$ multiplied by the first term in the right-hand side of (5.6) is identically zero. Consequently, we get

$$\begin{aligned} f^*(x) - f(x) &= h^{2m} \sum_{|\alpha|_1=2m} \int_{\mathbb{R}^d} \psi_\rho(x/h - t)(t - x/h)^\alpha \\ &\quad \times \int_0^1 2m(1-y)^{2m-1} D^\alpha f((1-y)x + yht)dy\,dt/\alpha!. \end{aligned}$$

Moreover, (5.2) (b) implies, via a change of variables, that

$$|f(x) - f^*(x)| \le c_\rho h^{2m} \sum_{|\alpha|_1=2m} \int_{\mathbb{R}^d} (1 + |t|)^{-d-k} \int_0^1 D^\alpha f(x - yht)dy\,dt/\alpha!$$

with $k > 0$. Consequently, by using the generalized Minkowski inequality, we obtain the required result:

$$\|f - f^*\|_{L_p(\Omega)} \le c_\rho h^{2m} |f|_{2m,L_p(\Omega)}, \quad 1 \le p \le \infty. \qquad \square$$

The next lemma describes the behavior of $f^* - s_{f^*,X}$.

Lemma 5.3. *Let f^* be defined as in (5.4) and $s_{f^*,X}$ the interpolant to f^* on X using the basis function ϕ_λ. Suppose that that the parameter λ satisfies the relation $\lambda = \rho h$ for some $\rho > 0$. Then there is a constant $c > 0$ such that for every function $f \in V_2^{2m}(\Omega)$,*

$$\|f^* - s_{f^*,X}\|_{L_p(\Omega)} \le ch^{m+\gamma_p}|f|_{2m,L_2(\Omega)}, \quad 1 \le p \le \infty,$$

where $\gamma_p = \min(m - d/2 + d/p, m)$ as in (1.8).

Proof. It is clear from (5.1) that $\hat{\psi}_\rho(\theta) = \tau(\theta)\hat{\phi}_\rho(\theta)$, where τ is a trigonometric function of the form

$$\tau(\theta) := \sum_{\alpha \in \mathcal{G}} \mu(\alpha) e^{-i\alpha\cdot\theta}.$$

Then, invoking (2.1) and applying the relation $\rho = \lambda/h$, we deduce that $\hat{\phi}_\rho(h\theta) = h^{-2m}\hat{\phi}_\lambda(\theta)$. Hence, the explicit form of f^* in (5.4) implies that

$$\hat{f}^* = \hat{\psi}_\rho(h\cdot)\hat{f} = h^{-2m}\tau(h\cdot)\hat{\phi}_\lambda\hat{f}. \tag{5.7}$$

Considering the decaying condition of ψ_ρ in (5.1), we realize that $\hat{\psi}_\rho$ is continuous everywhere, in particular, at the origin. In addition, since $\hat{\psi}_\rho(0) = 1$, the trigonometric function τ should have a zero of order $2m$ at the origin. Thus, it is obvious that the function f^* belongs to the space $\mathcal{F}_\phi$. Consequently, a direct application of Corollary 3.5 yields the inequality

$$\|f^* - s_{f^*,X}\|_{L_p(\Omega)} \le ch^{\gamma_p}|f^* - s_{f^*,X}|_{\phi_\lambda}, \tag{5.8}$$

with γ_p being defined via (1.8). Using the fact that $(f^* - s_{f^*,X}, s_{f^*,X})_{\phi_\lambda} = 0$ (see (2.5)), we see that

$$\begin{aligned} |f^* - s_{f^*,X}|^2_{\phi_\lambda} &= (f^* - s_{f^*,X}, f^* - s_{f^*,X})_{\phi_\lambda} \\ &= (f^*, f^* - s_{f^*,X})_{\phi_\lambda} \\ &= \int_{\mathbb{R}^d} (\hat{f}^*/\hat{\phi_\lambda})(t)(f^* - s_{f^*,X})^\wedge(t)dt \\ &= \int_{\mathbb{R}^d} (\hat{f}^*/\hat{\phi_\lambda})^\vee(t)(f^* - s_{f^*,X})(t)dt, \end{aligned} \tag{5.9}$$

where the last equation is valid by the Parseval's identity. Now, it is immediate from (5.7) that

$$\hat{f}^*/\hat{\phi}_\lambda = h^{-2m}\tau(h\cdot)\hat{f}. \tag{5.10}$$

According to Lemma 2.2.13 in [DR], $\sum_{\alpha\in\mathcal{G}} \mu(\alpha)\alpha^\beta = 0$ for any $|\beta|_1 < 2m$. Hence, expanding $\tau(h\vartheta)$ in a Taylor series about $\vartheta = 0$, we have

$$\begin{aligned} \tau(h\theta) &= \sum_{\alpha\in\mathcal{G}} \mu(\alpha)(ih\alpha\cdot\theta)^{2m} e^{-iyh(\alpha\cdot\theta)}/(2m)! \\ &= (ih)^{2m} \sum_{\alpha\in\mathcal{G}} \mu(\alpha) \sum_{|\beta|_1 = 2m} c_\beta \alpha^\beta \theta^\beta e^{-iyh(\alpha\cdot\theta)}/(2m)!, \end{aligned} \tag{5.11}$$

where the c_β's are suitable constants and $y \in [0, 1]$. Therefore, combining (5.10) and (5.11), we obtain the equation

$$(\hat{f}^*/\hat{\phi}_\lambda)^\vee = \sum_{\alpha\in\mathcal{G}} \mu(\alpha) \sum_{|\beta|_1=2m} c_\beta\alpha^\beta D^\beta f(\cdot - yh\alpha)/(2m)!. \tag{5.12}$$

Here, since $f \in V_p^{2m}(\Omega)$, the function $D^\beta f$ with $|\beta|_1 = 2m$ is supported in Ω. Further, since $y \in [0,1]$, it is evident that the function $D^\beta f(\cdot - yh\alpha)$, $\alpha \in \mathcal{G}$, is supported in the domain

$$\Omega_h := \bigcup_{\alpha \in \mathcal{G}} (\Omega + \alpha h).$$

Then, using (5.12), (5.9) yields the following estimates:

$$\begin{aligned} (5.13)\ |f^* - s_{f^*,X}|^2_{\phi_\lambda} &\leq c \sum_{|\beta|_1=2m} \sum_{\alpha \in \mathcal{G}} \int_{\Omega_h} |D^\beta f(t - yh\alpha)(f^* - s_{f^*,X})(t)| dt \\ &\leq c' \, |f|_{2m,L_2(\mathbb{R}^d)} \|f^* - s_{f^*,X}\|_{L_2(\Omega_h)}. \end{aligned}$$

Recalling that $\mathcal{G}$ is a finite subset of $\delta\mathbb{Z}^d$ ($\delta > 0$), we find that the density of X in Ω_h (not Ω) is bounded by ch with a constant c depending only on $\mathcal{G}$. Thus, a direct application of Corollary 3.5 with $p = 2$ yields

$$\|f^* - s_{f^*,X}\|_{L_2(\Omega_h)} \leq ch^m |f^* - s_{f^*,X}|_{\phi_\lambda}.$$

Combining this inequality with (5.13) yields the estimate

$$|f^* - s_{f^*,X}|_{\phi_\lambda} \leq ch^m |f|_{2m,L_2(\mathbb{R}^d)}.$$

Since $f \in V_2^{2m}(\Omega)$, we have $|f|_{2m,L_2(\mathbb{R}^d)} = |f|_{2m,L_2(\Omega)}$. Therefore, applying the above inequality to (5.8), we obtain the lemma's claim. □

Lemma 5.4. *Let f^* be defined as in* (5.4)*. Then there is a positive constant c such that for every function $f \in V_\infty^{2m}(\Omega)$, we have*

$$\|s_{f-f^*,X}\|_{L_p(\Omega)} \leq ch^{m+\gamma_p} |f|_{2m,L_\infty(\Omega)}.$$

Proof. The proof technique of Lemma 4.5 is adapted in a straightforward fashion to prove this lemma. Hence the proof is only sketched here. Using the Gaussian interpolant $g_{f,X}$ in (4.5), we can write $s_{f-f^*,X}$ as follows:

$$(5.14)\quad \|s_{f-f^*,X}\|_{L_p(\Omega)} \leq \|s_{f-f^*,X} - g_{f-f^*,X}\|_{L_p(\Omega)} + \|g_{f-f^*,X}\|_{L_p(\Omega)}.$$

It is immediate from Lemma 4.3 and Lemma 5.2 that

$$\|g_{f-f^*,X}\|_{L_p(\Omega)} \leq c\|f - f^*\|_{L_\infty(\Omega)} \leq ch^{2m} |f|_{2m,L_\infty(\Omega)}.$$

Hence, it remains to estimate only the first term on the right-hand side in (5.14). Since $s_{f-f^*,X} = s_{g_{f-f^*,X},X}$, we apply Corollary 3.5 and Lemma 4.3 to get

$$\begin{aligned} \|s_{f-f^*,X} - g_{f-f^*,X}\|_{L_p(\Omega)} &\leq ch^{\gamma_p} |g_{f-f^*,X}|_{\phi_\lambda} \\ &\leq ch^{\gamma_p - m} \|f - f^*\|_{L_\infty(\Omega)} \\ &\leq ch^{\gamma_p + m} |f|_{2m,L_\infty(\Omega)}, \end{aligned}$$

the last inequality being a consequence of Lemma 5.2. □

We summarize the results of this section:

Theorem 5.5. *Let $s_{f,X}$ be as in* (1.1)*, and let ϕ_λ be the "shifted" surface spline. Assume that the parameter λ in ϕ_λ is chosen to be $\lambda = \rho h$ for some $\rho > 0$. Let $\gamma_p = \min(m, m - d/2 + d/p)$ with $1 \leq p \leq \infty$. Then, for every $f \in V_\infty^{2m}(\Omega)$,*

$$\|f - s_{f,X}\|_{L_p(\Omega)} \leq c_f h^{m+\gamma_p},$$

where the constant c_f depends on f.

Proof. It is obvious that $V_\infty^{2m}(\Omega) \subset V_p^{2m}(\Omega)$ for $1 \le p < \infty$. Therefore, the desired result is immediate from Lemmas 5.2–5.4. □

Note that when $1 \le p \le 2$, we obtained the approximation order $O(h^{2m})$, which is equivalent to the case that Ω is all of $\mathbb{R}^d$ and X is an infinite uniform grid. In particular, as we observed before, when the parameter λ in ϕ_λ is zero, it becomes the surface spline ϕ_0. In this case, we do not have to employ the function f^*, but can estimate the error $f - s_{f,X}$ directly by using the technique of Lemma 5.3. Then we have the following estimate:

Corollary 5.6. *Let ϕ_0 be the surface spline function. Let f be a function in the space $V_2^{2m}(\Omega)$. Then we have an error bound of the form*

$$\|f - s_{f,X}\|_{L_p(\Omega)} \le c_f h^{m+\gamma_p}$$

with γ_p in (1.8).

Remark. The improvement of error bounds of the radial basis function interpolation method was first obtained by Schaback in [S3], where he considered the case that the Fourier transform of a basis function does not decay faster than $|\cdot|^{-k}$ for some $k > 0$. However, the conditions of this paper are more general; when $\lambda > 0$, the Fourier transform $\hat{\phi}_\lambda$ (2.1) decays exponentially. Also, when $\lambda = 0$, the function $\hat{\phi}_0$ decays like $|\cdot|^{-2m}$.

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