Convergence Property of Increasingly Flat Radial Basis function Interpolation to Polynomial Interpolation

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September 16, 2005

Abstract

In this paper, multivariate interpolation by smooth radial basis function is considered. We prove that when the basis function is scaled to be increasingly flat, multivariate interpolants by smooth radial basis functions converge to multivariate polynomial interpolants. Moreover, from this view point, we discuss the difference between the well-known radial functions and the Bessel function. To this end, all the radial basis function are categorized into two groups: positive definite functions and conditionally positive definite functions of order $m > 0$. First, in the case of positive definite radial function, we prove the sufficient criteria in [9] for the convergence of radial function interpolants. Next, we extend the results to the case of conditionally positive definite radial functions of order $m > 0$.

AMS (MOS) Subject Classification. 41A05, 41A15, 41A25, 41A30, 41A63
Keywords: Radial basis function, Interpolation, Polynomial, Conditionally Positive Definite Function.

1 Introduction

One of the most important issue in many area of applied mathematics and applications is to construct an approximation to scattered data: Given a set of scattered points X in $\Omega \subset \mathbb{R}^d$ and values $f|_X$ of some underlying function f, the goal is to find a function $s : \Omega \to \mathbb{R}^d$ such that s approximates f in some sense. Radial basis functions (RBFs) provide well-established tools for solving the scattered data approximation or interpolation problem. In addition, these days, they are increasingly being popular for the numerical solution of partial differential equations.

A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radial in the sense that $\phi(x) = \Phi(|x|)$ where $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$ stands for the usual Euclidean norm. The common choices of ϕ can be divided into two groups, piecewise smooth such as thin plate splines $\phi(x) = |x|^2 \log |x|$ and infinitely smooth such as

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multiquadrics $\phi(x) = (|x|^2 + \lambda^2)^{1/2}$. In this paper, we are interested in the infinitely smooth basis function ϕ (especially with a tension parameter) so that it can be expanded as

$$\phi(x) = \Phi(|x|) = \sum_{n=0}^{\infty} c_n |x|^{2n},$$

which actually means ϕ is analytic. Further, the function ϕ is considered as a tempered distribution, and we assume that its Fourier transform $\hat{\phi}$ is of the form

$$|\cdot|^{2m}\hat{\phi} = F \in L_\infty(\mathbb{R}^d), \quad m \geq 0, \tag{1.1}$$

where, indeed, the function F is nonnegative on $\mathbb{R}^d$ and positive at least on an open subset of $\mathbb{R}^d$.

An attractive feature of using infinitely smooth function is that it can provide the spectral approximation order. Typical examples of such basis functions are given as follows: For $d,\ m \in \mathbb{N}$ and $\lambda > 0$,

- $\phi(x) \ := \ (|x|^2 + \lambda^2)^{m-d/2}$, d odd, $m > d/2$, (multiquadrics),
- $\phi(x) \ := \ (|x|^2 + \lambda^2)^{m-d/2} \log(|x|^2 + \lambda^2)^{1/2}, m > d/2, d$ even, ('shifted$'$ surface splines).
- $\phi(x) \ := \ (|x|^2 + \lambda^2)^{m-d/2}, 0 < m < d/2,$ (inverse multiquadrics),
- $\phi(x) \ := \ e^{-|x|^2/\lambda^2}$, (Gaussians).

The RBFs can be scaled in a way of being wider by a shape parameter ϵ, i.e., $\phi_\epsilon(x) := \phi(\epsilon x)$, which is indeed equivalent to taking a large λ in the above basis functions. It has been observed in [3, 8] that for smooth data, a very small value of ϵ gives very accurate results both for interpolation problems and when solving elliptic partial differential equations. In this case, the basis function becomes very flat and the condition number of the interpolation system grows rapidly. However, the limit RBF interpolant is often well behaved so that it converges to a finite number.

In [2], Driscoll and Fornberg introduced a surprising observation that the limit of RBF interpolant is a (multivariate) finite order polynomial interpolant, if it exists. Recently, a study in the same direction has been done independently by Schaback [13]. He showed that interpolation with scaled Gaussian always converges to the de Boor-Ron polynomial interpolant when the Gaussian widths increase. Subsequent to carrying out this study, we became aware of another recent paper by Fornberg and Larsson [9]; they generated a matrix (2.3) (hereafter, called 'FL-matrix') from the Taylor expansion of the given RBF, and proved that the non-singularity of the FL-matrix guarantees the convergence of the RBF interpolation to a multivariate polynomial interpolation as the shape parameter $\epsilon \to 0$. However, the non-singularity of the FL-matrix is just conjectured and yet to be proved. Moreover, it is also observed in [9] that this convergence property holds for the commonly well-known RBF interpolants but some exceptional behavior happened for the Bessel RBF. (see [9, Example 2.4]). There needs to be a clear discussion on the reason for the deviant behavior of the Bessel RBF interpolant. Thus, the aim of this paper is to provides the detailed proof and conditions which addresses all the above issues. To this end, we categorize the (smooth) RBFs into two groups: positive definite functions and conditionally positive definite functions of order $m > 0$. Then, the specific contribution of this work is given as follows:

- The FL-matrix is proved to be non-singular for all positive definite RBFs. It implies that a positive definite RBF interpolation converges to a multivariate polynomial interpolation as the shape parameter $\epsilon \to 0$.

- We provide more detailed discussion on the difference between the commonly used RBFs and the Bessel RBF. It verifies that the FL-matrix for the Bessel RBF is singular (as conjectured by Fornberg and Larsson in [9]).
- The FL-matrix is modified for conditionally positive definite RBFs of order $m > 0$. The FL-matrix is proved to be non-singular, which verifies that increasing flat RBF interpolants tend to polynomial interpolants.
- The state-of-the-art in the studies on the limit of increasing flat RBF have considered the interpolation from the space $\text{span}\{\phi(\varepsilon(x - x_j)) : x_j \in X\}$ [8, 9, 13]. However, usually, when ϕ is conditionally positive definite of order m, some suitable polynomial of degree $m - 1$ are added for the construction of interpolation. Thus, our next goal is concerned with RBF interpolant from the augmented space $\text{span}\{\phi(\varepsilon(x - x_j)) : x_j \in X\} + \Pi_{<m}$ and prove its convergence to polynomial interpolation as $\varepsilon \to 0$.

We use the following notation throughout this paper. When $\mathbf{g}$ is a matrix or a vector, $\|\mathbf{g}\|_p$ indicates its p-norm with $1 \leq p \leq \infty$. For $\alpha, \beta \in \mathbb{Z}_+^d := \{(\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}^d : \gamma_k \geq 0\}$, we set $\alpha! := \alpha_1! \cdots \alpha_d!$, $|\alpha|_1 := \sum_{k=1}^d \alpha_k$, and $\alpha^\beta = \alpha_1^{\beta_1} \cdots \alpha_d^{\beta_d}$. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t) \exp(-i\theta \cdot t)\, dt.$$

Also, for a function $f \in L_1(\mathbb{R}^d)$, we use the notation $f^\vee$ for the inverse Fourier transform. The Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$. Let $\Pi_{<m}$ denote the space of d-variate algebraic polynomials of degree $< m$ on $\mathbb{R}^d$ and denote

$$N_m = dim\Pi_{<m+1} = \binom{m+d}{d}. \tag{1.2}$$

2 The Fornberg and Larsson Conjecture

In this section, we revisit the work of Fornberg and Larsson in [9] and introduce their conjecture. For this, the multi-index definitions from [9] are introduced.

Definition 2.1 *Let $\alpha =: (\alpha_1, \ldots, \alpha_d)$ and $\beta =: (\beta_1, \ldots, \beta_d)$ be in $\mathbb{Z}_+^d$. Then we define the following terminologies:*

(a) The multi-indices α, β are said to have the same parity if the components α_j and β_j with $j = 1, \ldots, d$ have the same parity.

(b) The polynomial ordering of a sequence of multi-indices is determined in the way that the index α comes before β if $|\alpha|_1 < |\beta|_1$, or if $|\alpha|_1 = |\beta|_1$, $\alpha_j = \beta_j$, $j = 1, \ldots, p$ and $\alpha_{p+1} > \beta_{p+1}$. **(Should it be $\alpha_{p+1} < \beta_{p+1}$?)**

Definition 2.2 *Let I_K, where $K \geq 0$, be the polynomially ordered sequence of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $|\alpha|_1 \leq K$. For $n \in \mathbb{N}$, $I_K(n)$ denotes the n-th multi-index in the sequence.*

Definition 2.3 *Let $I_{p,K}$, where $0 \le p \le d$ and $K \ge 0$, be the polynomially ordered sequence of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha|_1 \le K$ such that $\alpha_1, \ldots, \alpha_p$ are odd numbers, the others are even numbers. For $n \in \mathbb{N}$, $I_{p,K}(n)$ denotes the n-th multi-index in the sequence.*

Definition 2.4 *Let I_{2n}^{α} be the polynomially ordered set of all multi-indices β such that $|\alpha+\beta|_1 = 2n$, and α and β have the same parity.*

Let $X = \{x_j : j = 1, \ldots, N\}$ and $N_{k-1} < N \le N_k$ with N_K in (1.2). Let $p_1, \ldots, p_N$ be N independent polynomials in $\Pi_{<K+1}$. We say X is unisolvent with respect to $\{p_j : j = 1, \ldots, N\}$ if there is a unique linear combination $\sum \beta_j p_j(x)$ such that it interpolates the data over the point set X. With this definition at hand, we introduce the relation between polynomial interpolation and the distribution of the set X. Specifically, we can classify the polynomial interpolants as follows:

(I) If $N = N_{K,d}$ and the set X is unisolvent with respect to any set of N linearly independent polynomials in $\Pi_{<K+1}$, then there is a unique interpolating polynomial of degree K on the point set X.

(II) If $N < N_{K,d}$ and the set X is unisolvent with respect to any set of N linearly independent polynomials in $\Pi_{<K+1}$, then there is an interpolating polynomial of degree K on the point set X.

(III) If X is non-unisolvent and the degree of the minimal non-degenerate basis is M, then there is an interpolating polynomial of degree M on the point set, for each choice of a minimal non-degenerate basis.

Invoking that the RBFs ϕ considered in this paper are real analytic, our analysis is based on the Taylor expansion of $\phi(x - x_j) = \Phi(|x - x_j|)$ with respect to $|x - x_j|$ as follows:

$$\phi(x - x_j) = \sum_{n=0}^{\infty} c_n |x - x_j|^{2n}. \tag{2.1}$$

Here, the coefficient of x^α in the expansion of $|x - x_j|^{2n}$ can be written by

$$|x - x_j|^{2n}\big|_{x^\alpha} = \sum_{\beta \in I_{2n}^{\alpha}} (-1)^{|\alpha|_1} \frac{n!}{(\frac{\alpha+\beta}{2})!} \frac{(\alpha+\beta)!}{\alpha!\beta!} x_j^{\beta}$$

Combining this with (2.1), we see that the coefficient of x^α in the expansion of $\phi(x - x_j)$ is

$$\phi(x - x_j)|_{x^\alpha} = \sum_{n=|\lfloor \frac{\alpha+\vec{1}}{2} \rfloor|_1}^{\infty} c_n \sum_{\beta \in I_{2n}^{\alpha}} (-1)^{|\alpha|_1} \frac{n!}{(\frac{\alpha+\beta}{2})!} \frac{(\alpha+\beta)!}{\alpha!\beta!} x_j^{\beta}, \tag{2.2}$$

where $\vec{1} = (1, \ldots, 1) \in \mathbb{R}^d$ and for any $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{Z}_+^d$, $\lfloor \gamma \rfloor =: (\lfloor \gamma_1 \rfloor, \ldots, \lfloor \gamma_d \rfloor)$ with $\lfloor s \rfloor$ the greatest integer less than or equal to s.

Now, we define the matrix $\mathbf{A}_{p,K}$ corresponding to the polynomially ordered sequence $I_{p,K}$ by

$$\mathbf{A}_{p,K}(k, \ell) = c_n (-1)^{|\alpha|_1} \frac{n!}{(\frac{\alpha+\beta}{2})!} \frac{(\alpha+\beta)!}{\alpha!\beta!}, \tag{2.3}$$

where $\alpha := I_{p,K}(k)$, $\beta := I_{p,K}(\ell)$, and $2n = |\alpha+\beta|_1$. Hereafter, the matrix $\mathbf{A}_{p,K}$ is called FL-matrix.

Fornberg-Larsson's Conjecture: Let $0 \le p \le d$ and $0 \le J \le K$. Then the FL-matrices $\mathbf{A}_{p,K}$ are non-singular for all commonly used RBFs such as (inverse) multiquadrics and Gaussians.

Theorem 2.5 *([9]) Let $X = \{x_1, \dots, x_N\}$ and suppose that the Fornberg-Larsson Criteria hold. Then, as the shape parameter $\varepsilon \to 0$, the limit of the RBF interpolant is an interpolating polynomial, say $P(x)$, of a certain degree to the given data. The properties of the interpolating polynomial are given as follows:*

(a) *If the set X is of type (I), then the limit is the unique interpolating polynomial of degree K to the given data.*

(b) *If the set X is of type (II), the limit is an interpolanting polynomial of degree K to the given data. The exact polynomial depends on the choice of RBFs.*

(c) *If the set X is of type (III) and if the limit exists, the limit is an interpolating polynomial of degree M.*

In the following section, we will prove that the results (a), (b) and (c) in Theorem 2.5 are true for all well-known RBFs. However, there happened some exceptional behavior for the case of the Bessel RBF. The difference between the commonly used RBFs and Bessel RBF will be discussed. Furthermore, we extends Theorem 2.5 to the case of RBF interpolation from the augmented space $\mathrm{span}\{\phi(x-x_j) : x_j \in X\} + \Pi_{<m}$.

3 The Main Results

3.1 Positive Definite Function

In this section, we prove the Fornberg-Larsson's conjecture is true for positive definite functions. To this end, we obtain a different expression of the matrix $\mathbf{A}_{p,K}$ (2.3).

Definition 3.1 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $\Omega \subset \mathbb{R}^d$. We say that ϕ is conditionally positive definite of order $m \in \mathbb{Z}_+ := \{1, 2, \dots\}$ if for every finite set of pairwise distinct points $X = \{x_1, \dots, x_N\} \subset \Omega$ and for every $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N \setminus 0$ satisfying*

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \quad p \in \Pi_{<m}, \tag{3.1}$$

the quadric form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i - x_j)$$

is positive definite. A conditionally positive definite function of order 0 is called a positive definite function.

Indeed, the condition (3.1) may seem a bit technical, and hard to verify in practice. However, the positive definiteness of continuous and absolutely integrable functions ϕ is equivalent to the Fourier transform $\hat{\phi}$ being nonnegative on $\mathbb{R}^d$ and positive at least on an open subset of $\mathbb{R}^d$. We note in passing that this argument is a consequence of the simple identity

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k\phi(x_j-x_k)=\int_{\mathbb{R}^d}\hat{\phi}(\theta)\Big|\sum_{j=1}^{N}\alpha_j e^{ix_j\cdot\theta}\Big|^2 d\theta \tag{3.2}$$

for any $(\alpha_1,\ldots,\alpha_N)\in\mathbb{R}^N\setminus 0$ and the fact that the map $\theta\mapsto\sum_{j=1}^{N}\alpha_j e^{ix_j\cdot\theta}$, $\theta\in\mathbb{R}^d$, has zeros at most on a set of measure zero. This identity is also generalized to the case of conditionally positive definite functions ϕ of order $m>0$.

Suppose that a continuous function $f:\mathbb{R}^d\to\mathbb{R}$ is known only at a set of discrete points $X:=\{x_1,\ldots,x_N\}$ in $\Omega\subset\mathbb{R}^d$. Radial basis function interpolant of the data $(x_j,f(x_j))$, $j=1,\ldots,N$, with positive definite function ϕ is given by

$$s(x,\varepsilon):=\sum_{j=1}^{N}a_j\phi(\varepsilon(x-x_j)), \tag{3.3}$$

where the coefficients a_j $(j=1,\ldots,N)$ are obtained by solving the linear system

$$s(x_j,\varepsilon)\;=\;f(x_j),\quad j=1,\ldots,N. \tag{3.4}$$

The unique solution of the above linear system is guaranteed when the function ϕ is positive definite [7].

Lemma 3.2 *Let the matrix $\mathbf{A}_{p,K}$ be defined as in (2.3). Then, if $\alpha=I_{p,K}(k),\beta=I_{p,K}(\ell)$ with $\alpha,\beta\in\mathbb{Z}_+^d$ and $2n=|\alpha+\beta|_1$, we have*

$$\mathbf{A}_{p,K}(k,\ell)\;=\;(-1)^{|\alpha|_1}\frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!}.$$

Proof. Since ϕ is radially symmetric, $\phi(\cdot-x_j)=\phi(x_j-\cdot)$. Taking the Taylor expansion of $\phi(x_j-\cdot)$ around x_j yields the expression

$$\begin{aligned}\phi(x_j-x)\;&=\;\sum_{|\alpha|_1=0}^{\infty}\phi^{(\alpha)}(x_j)\frac{(-x)^\alpha}{\alpha!}\\ &=\;\sum_{|\alpha|_1=0}^{\infty}\Bigg(\sum_{|\beta|_1=0}^{\infty}\phi^{(\alpha+\beta)}(0)\frac{x_j^\beta}{\beta!}\Bigg)\frac{(-x)^\alpha}{\alpha!},\end{aligned}$$

the second identity being a consequence of the Taylor expansion of $\phi^{(\alpha)}(x_j)$ around the origin. Then the coefficient of x^α in this expansion of $\phi(x_j-x)$ is denoted by

$$\begin{aligned}\phi(x-x_j)|_{x^\alpha}\;&=\;\sum_{|\beta|_1=0}^{\infty}(-1)^{|\alpha|_1}\frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!}x_j^\beta \\ &=\;\sum_{n=|\lfloor\frac{\alpha+\vec{1}}{2}\rfloor|_1}^{\infty}\;\sum_{\beta\in I_{2n}^{\alpha}}(-1)^{|\alpha|_1}\frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!}x_j^\beta.\end{aligned} \tag{3.5}$$

Comparing (3.5) with (2.2), we obtain from the definition of $\mathbf{A}_{p,K}$ in (2.3) that

$$\mathbf{A}_{p,K}(k,\ell) \quad = \quad (-1)^{|\alpha|_1} \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!},$$

where $\alpha = I_{p,K}(k)$ and $\beta = I_{p,K}(\ell)$ with $\alpha, \beta \in \mathbb{Z}_+^d$. $\square$

Due to the above lemma, the non-singularity of each matrix $\mathbf{A}_{p,K}$ can be shown by proving that the matrix, whose components are of the form $\phi^{(\alpha+\beta)}(0)/\alpha!\beta!$, is non-singular.

Lemma 3.3 *Let ϕ be a positive definite function and let its Fourier transform $\hat{\phi}$ be positive on an open set in $\mathbb{R}^d$. For any integer $K > 0$, define the matrix $\mathbf{T}_\phi$ by*

$$\mathbf{T}_\phi = \left(\frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} : 0 \leq |\alpha|_1, |\beta|_1 \leq K \right). \tag{3.6}$$

Then $\mathbf{T}_\phi$ is non-singular. $\square$

Proof. Note that T_ϕ is an $N_K \times N_K$ matrix. Let $\gamma =: (\gamma_\alpha : |\alpha|_1 \leq K)$ be an arbitrary non-zero vector in $\mathbb{R}^{N_K}$. Then the non-singularity of $\mathbf{T}_\phi$ is guaranteed by showing that $\gamma \mathbf{T}_\phi \gamma^T > 0$, i.e.,

$$\sum_{|\beta|_1 \leq K} \sum_{|\alpha|_1 \leq K} \gamma_\alpha \gamma_\beta \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} > 0. \tag{3.7}$$

It is easy to see the identity

$$\sum_{|\beta|_1 \leq K} \sum_{|\alpha|_1 \leq K} \gamma_\alpha \gamma_\beta \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} = \int_{\mathbb{R}^d} \hat{\phi}(\theta) \left| \sum_{|\alpha|_1 \leq K} \gamma_\alpha \frac{\theta^\alpha}{\alpha!} \right|^2 d\theta.$$

Since the Fourier transform $\hat{\phi}$ is nonnegative on $\mathbb{R}^d$ and positive at least on an open subset of $\mathbb{R}^d$, the proof is completed. $\square$

We now prove Fornberg-Larsson's conjecture.

Theorem 3.4 *Let ϕ be a positive definite function and let its Fourier transform $\hat{\phi}$ be positive on an open set in $\mathbb{R}^d$. Then, for any integer $p, K \geq 0$, the matrix $\mathbf{A}_{p,K}$ (2.3) is non-singular.*

Proof. For an arbitrarily given $\gamma = (\gamma_\alpha : \alpha \in I_{p,K})$, define $\tilde{\gamma} = (\tilde{\gamma}_\alpha : 0 \leq |\alpha|_1 \leq K)$ by

$$\tilde{\gamma}_\alpha := \begin{cases} \gamma_\alpha, & \text{if} \quad \alpha \in I_{p,K}, \\ 0, & \text{otherwise}. \end{cases}$$

Then we have

$$\sum_{\beta \in I_{p,K}} \sum_{\alpha \in I_{p,K}} \gamma_\alpha \gamma_\beta \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} = \sum_{|\beta|_1 \leq K} \sum_{|\alpha|_1 \leq K} \tilde{\gamma}_\alpha \tilde{\gamma}_\beta \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} = \tilde{\gamma} \mathbf{T}_\phi \tilde{\gamma}^T,$$

for $\mathbf{T}_\phi$ as given in (3.6). Now, the theorem follows directly from Lemmas 3.3 and 3.2. $\square$

As a conclusion, we can get the following results, which are actually re-statements of Theorem 2.5.

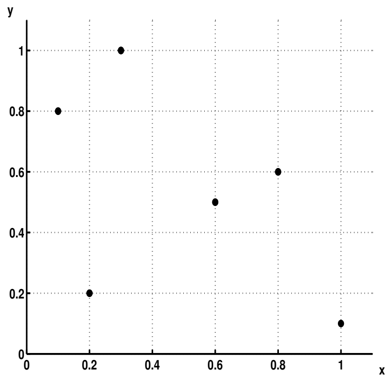

figure 1: Scattered Points X.

Theorem 3.5 *Let $X = \{x_1, \ldots, x_N\}$ and assume that ϕ is a positive definite function and its Fourier transform $\hat{\phi}$ is positive on an open set in $\mathbb{R}^d$. Then, as the shape parameter $\varepsilon \to 0$, the limit of the RBF interpolant is an interpolation polynomial, say $P(x)$, of a certain degree to the given data. The properties of the polynomial interpolants are given as follows:*

(a) If the set X is of type (I), then the limit is the unique interpolation polynomial of degree K to the given data.

(b) If the set X is of type (II), the limit is an interpolanting polynomial of degree K to the given data. The exact polynomial depends on the choice of RBFs.

(c) If the set X is of type (III) and if the limit exists, the limit is a polynomial of degree M that interpolates the given data.

An interesting example was obsereved in [9, Example 2.4] that all RBFs except Bessel radial function have the same limit, which is the unique polynomial interpolant, as the shape parameter tends to zero. It can be described briefly as follows:

Example 3.6 ([9, Example 2.4]) *Let X be a set of six points as in figure 1 and $f(x_j) = \delta_{0,j}$ with $j = 1, \ldots, 6$. These six points do not have any particular pattern. Then all the limit of the known RBF interpolants converge to the same two-variable polynomial of degree 2, which is the unique polynomial interpolant. However, Bessel radial function gives a different limit, which is a third order polynomial.*

The Bessel radial function $\phi(x) := J_0(|x|)$ is known to be positive definite ([11, Theorem 3.1]) but its Fourier transform is proportional to dirac distribution, i.e., $\hat{\phi} = c\delta(\cdot - 1)$. It has been conjectured by Fornberg and Lasson in [8] that the reason for this deviant behavior is that the Bessel function does not fulfill the non-singularity condition in the Fornberg-Larsson Criteria. The following corollary proves it.

Corollary 3.7 *Let $\phi(x) := J_0(|x|)$ be the Bessel RBF. For any integer $K \geq 2$, the matrix $\mathbf{T}_\phi$ in (3.6) is not positive-definite.*

Proof. Recall the Hankel transform on $\mathbb{R}^d$:

$$J_0(|x|) = \frac{1}{2\pi}\int_{|\theta|=1} e^{x\cdot\theta}d\theta, \quad x \in \mathbb{R}^d. \tag{3.8}$$

Then, for any $\gamma = (\gamma_\alpha : |\alpha|_1 \leq K)$, we can easily get the identity

$$\sum_{|\beta|_1\leq K}\sum_{|\alpha|_1\leq K}\gamma_\alpha\gamma_\beta\frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} = \int_{|\theta|=1}\left|\sum_{|\alpha|_1\leq K}\gamma_\alpha\frac{\theta^\alpha}{\alpha!}\right|^2 d\theta. \tag{3.9}$$

Choose γ_α with $|\alpha|_1 \leq K$ such that

$$\sum_{|\alpha|_1\leq K}\gamma_\alpha\frac{\theta^\alpha}{\alpha!} = 1-(\theta_1^2+\cdots+\theta_d^2), \qquad \theta =: (\theta_1,\ldots,\theta_d).$$

Then the above term in (3.9) becomes zero, which proves that the matrix $\mathbf{T}_\phi$ is not positive-definite. □

3.2 Conditionally Positive Definite Function

The RBF interpolant to the data $(x_j, f(x_j))$, $x_j \in X = \{x_1,\ldots,x_N\}$, with a conditionally positive definite function ϕ of order m is given by

$$s(x,\varepsilon) := \sum_{j=1}^{N} a_j\phi(\varepsilon(x-x_j)) + \sum_{i=1}^{N_{m-1}} b_i p_i(x), \tag{3.10}$$

where $p_1,\ldots,p_{N_m}$ is a basis for $\Pi_{<m}$. The coefficients a_j $(j = 1,\ldots,N)$ and b_i $(i = 1,\ldots,N_{m-1})$ are obtained by solving the linear system which can be written in a matrix form as

$$\begin{pmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P^T} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \tag{3.11}$$

where $\mathbf{A}$ and $\mathbf{P}$ are the $N\times N$ and $N\times N_m$ matrices that have the elements $\mathbf{A}_{ij} = \phi(\varepsilon(x_i - x_j))$ and $\mathbf{P}_{ij} = p_j(x_i)$, respectively. Further, $\mathbf{a} \in \mathbb{R}^N$ and $\mathbf{b} \in \mathbb{R}^{N_m}$ are the vectors of coefficients of $s(\cdot,\varepsilon)$, and the components of $\mathbf{f}$ are the data $f(x_j)$, $j = 1,\ldots,N$. Here, for $m > 0$, we require X to have the nondegeneracy property for $\Pi_{<m}$, i.e.,

$$q(x_j) = 0, \quad 1\leq j\leq N \quad \text{for} \quad q\in\Pi_{<m} \quad \text{implies} \quad q = 0. \tag{3.12}$$

The unique solution of the above linear system is guaranteed when the function ϕ is conditionally positive definite of order m [7].

To state our work precisely, we need some definitions:

Definition 3.8 *Let $I_{p,m,K}$, where $0 \leq p \leq d$ and $K \geq 0$, be the polynomially ordered sequence of all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_d)$ such that $\alpha_1,\ldots,\alpha_p$ are odd numbers, the others are even numbers and $m \leq |\alpha|_1 \leq K$. For $n \in \mathbb{N}$, $I_{p,m,K}(n)$ denotes the n-th multi-index in the sequence.*

Here and in the sequel, assume that ϕ is conditionally positive definite of order $m > 0$. Recalling the Taylor expansion of ϕ, that is,

$$\phi(x - x_j) = \sum_{n=0}^{\infty} c_n |x - x_j|^{2n}, \tag{3.13}$$

define the matrix $\mathbf{A}_{p,m,K}$ corresponding to the polynomially ordered sequence $I_{p,m,K}$ by

$$\mathbf{A}_{p,m,K}(k,\ell) \quad = \quad c_n (-1)^{|\alpha|_1} \frac{n!}{(\frac{\alpha+\beta}{2})!} \frac{(\alpha+\beta)!}{\alpha!\beta!}, \tag{3.14}$$

where $\alpha := I_{p,m,K}(k), \beta := I_{p,m,K}(\ell)$, and $2k = |\alpha + \beta|_1$. In what follows, we will show that the matrix $\mathbf{A}_{p,m,K}$ is non-singular.

Theorem 3.9 *For any integer $p, m, K > 0$, the matrix $\mathbf{A}_{p,m,K}$ is non-singular.*

Proof. Let $\alpha = I_{p,m,K}(k), \beta = I_{p,m,K}(\ell)$ with $\alpha, \beta \in \mathbb{Z}_+^d$ and $|\alpha|_1, |\beta|_1 \geq m$. Then, from Lemma 3.2, we find

$$\mathbf{A}_{p,m,K}(k,\ell) \quad = \quad (-1)^{|\alpha|_1} \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!}.$$

Thus, it suffices to prove the non-singularity of the the matrix $\mathbf{T}_\phi^{[m]}$ defined by

$$\mathbf{T}_\phi^{[m]} = \left(\frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} : \alpha, \beta \in I_{p,m,K} \right).$$

For this, we show that that for any $\gamma =: (\gamma_\alpha : \alpha \in I_{p,m,K})$, $\gamma \mathbf{T}_\phi^{[m]} \gamma^T > 0$. Indeed, we note that this is a consequence of the following relation

$$\begin{aligned} \gamma \mathbf{T}_\phi^{[m]} \gamma^T \quad &= \quad \sum_{\beta \in I_{p,m,K}} \sum_{\alpha \in I_{p,m,K}} \gamma_\alpha \gamma_\beta \frac{\phi^{(\alpha+\beta)}(0)}{\alpha!\beta!} \\ &= \quad \int_{\mathbb{R}^d} \hat{\phi}(\theta) \Big| \sum_{\alpha \in I_{p,m,K}} \gamma_\alpha \theta^\alpha \Big|^2 d\theta. \end{aligned} \tag{3.15}$$

Since $|\cdot|^{2m} \hat{\phi} = F \geq 0$ and ϕ is real analytic, we deduce that the function F decays faster than any polynomial degree around ∞. Thus, the last integral in (3.15) makes sense and positive; see the line below (1.1). It completes the proof. $\square$

Now, we extends Theorem 2.5 to the case of conditionally positive definiteness RBFs.

Theorem 3.10 *Let $X = \{x_1, \ldots, x_N\}$ be a set of scattered points and $s(\cdot, \varepsilon)$ in (3.10) be the RBF interpolant to a function f on X using $\phi(\varepsilon\cdot)$. Then, as the shape parameter $\varepsilon \to 0$, we have the limit of the RBF interpolant $s(x, \varepsilon)$ as follows:*

(a) *If the set X is of type (I), the limit is the unique polynomial interpolant of degree K to the given data.*

(b) *If the set X is of type (II), the limit is an polynomial interpolant of degree K to the given data. The exact polynomial depends on the choice of RBFs.*

(c) *If the set X is of type (III), the limit is a polynomial of degree $2K - m$ that interpolates the given data, if it exists.*

Theorem 3.11 *Let $X = \{x_1, \ldots, x_N\}$ be a set of scattered points. Assume that the RBF interpolant to a function f on X is of the form*

$$s(x, \varepsilon) = \sum_{j=1}^{N} a_j \phi(\varepsilon(x - x_j))$$

and the interpolation matrix $\mathbf{A} = (\phi(\varepsilon(x_i - x_j)) : x_i, x_j \in X)$ is non-singular. In addition, the points X to be required to have the nondegeneracy property for $\Pi_{<m}$ if ϕ is a conditionally positive definite function of order m. Then, as the shape parameter $\varepsilon \to 0$, we have the same conclusion as in Theorem 3.10.

Example 3.12 *Let the RBF ϕ be chosen to be one of what follows:*

(a) $\phi(x) := (|x|^2 + \lambda^2)^{m-d/2}$, *$d$ odd, $m > d/2$, (multiquadrics),*

(b) $\phi(x) := (|x|^2 + \lambda^2)^{m-d/2} \log(|x|^2 + \lambda^2)^{1/2}$, *$m > d/2$, d even, ('shifted' surface splines).*

(c) $\phi(x) := (|x|^2 + \lambda^2)^{m-d/2}$, *$0 < m < d/2$, (inverse multiquadrics),*

where d, $m \in \mathbb{N}$ and $\lambda > 0$. The above basis functions ϕ grows polynomially for large argument. Then, in the sense of tempered distributions, ϕ has the following generalized Fourier transform (see [4]):

$$\hat{\phi}(\theta) = c_{m,d} |\theta|^{-2m} \tilde{K}_m(|\lambda\theta|) \tag{3.16}$$

where $c_{m,d}$ is a positive constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|)$ with $K_\nu(|t|)$ the modified Bessel function of order ν. From [1], we find that

$$\begin{aligned} &\tilde{K}_\nu(|t|) \in C^{2\nu-1}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \{0\}), \\ &\tilde{K}_\nu(|t|) > 0, \quad \text{and} \quad \tilde{K}_\nu(|t|) \approx e^{-|t|}(1 + |t|^{(\nu-1/2)}). \end{aligned} \tag{3.17}$$

4 Proofs of Theorem 3.10 and Theorem 3.11

The general technique for the proofs is similar to the method given in [9], in spite of that the interpolant has an additional polynomial term associated to the order of conditionally positive definiteness.

The RBF can be written as follows (if we include a shape parameter ε):

$$\phi_\varepsilon(x) := \phi(\varepsilon x) = \sum_{j=0}^{\infty} c_j \varepsilon^{2j} |x|^{2j}. \tag{4.1}$$

The coefficients a_j $(j = 1, \ldots, N)$ and b_i $(i = 1, \ldots, N_{m-1})$ are obtained by Cramer's rule, and they must be rational functions of ε^2. That is, there exists an integer $q \geq 0$ such that we can write

$$a_j =: \varepsilon^{-2K} \sum_{n=-q}^{\infty} \varepsilon^{2n} a_{j,n}, \qquad b_i =: \varepsilon^{-2K} \sum_{n=-q}^{\infty} \varepsilon^{2n} b_{i,n}, \tag{4.2}$$

where at least one of $(a_{j,-q} : j = 1, \ldots, N)$ and $(b_{j,-q} : j = 1, \ldots, N_m)$ is a non-zero vector. Now, for each $n \geq -q$ and $\beta \in \mathbb{Z}_+^d$, the discrete moments of $(a_{1,n}, \ldots, a_{N,n})^T$ is defined by

$$\sigma_n^{[\beta]} := \sum_{j=1}^{N} a_{j,n} x_j^{\beta}, \quad n = -q, -q+1, \ldots. \tag{4.3}$$

Then for the moment $\sigma_n^{[\beta]}$ with $|\beta|_1 \leq m-1$ and $n \geq -q$, we have the estimate:

Lemma 4.1 *Let $\beta \in \mathbb{Z}_+^d$ with $|\beta|_1 \leq m-1$. Then, $\sigma_n^{[\beta]} = 0$ for any $n \geq -q$.*

Proof. Multiplying by ε^{2n+2K} the both sides of (4.3) and summing over $n = -q, -q+1, \ldots,$ we have that for any $\varepsilon > 0$

$$\begin{aligned} \varepsilon^{-2K} \sum_{n=-q}^{\infty} \varepsilon^{2n} \sigma_n^{[\beta]} &= \varepsilon^{-2K} \sum_{j=1}^{N} \sum_{n=-q}^{\infty} \varepsilon^{2n} a_{j,n} x_j^{\beta} \\ &= \sum_{j=1}^{N} a_j x_j^{\beta} = 0, \end{aligned}$$

where the last identity is a simple consequence of (3.11). Thus, from this relation, we deduce the required result $\sigma_n^{[\beta]} = 0$ with $n \geq -q$. $\square$

Let S_N indicate the N-set of polynomially ordered multi-indices, i.e., $\#S_N = N$. Define the matrix $\mathbf{V}$ by

$$\mathbf{V} = (x_j^{\beta} : j = 1, \ldots, N, \ \beta \in S_N).$$

Then from (4.3), we have

$$\mathbf{s}_n = \mathbf{V}\, \mathbf{a}_n, \quad n = -q, -q+1, \ldots, \tag{4.4}$$

where $\mathbf{s}_n := (\sigma_n^{[\beta]} : \beta \in S_N)^T$ and $\mathbf{a}_n := (a_{j,n} : j = 1, \ldots, N)^T$. It implies that if $\mathbf{V}$ is non-singular, each moment $\sigma_n^{[\beta]}$ can be written as

$$\sigma_n^{[\beta]} = \sum_{j=1}^{N} a_{j,n} x_j^{\beta} = \mathbf{x}^{\beta} \mathbf{a}_n = \mathbf{x}^{\beta} \mathbf{V}^{-1} \mathbf{s}_n, \tag{4.5}$$

with $\mathbf{x}^{\beta} := (x_1^{\beta}, \ldots, x_N^{\beta})$. This relation will be used later for the proof of Theorem 3.10 (b).

Recall that the RBF interpolant with a conditionally positive definite function ϕ of order m is given by

$$s(x, \varepsilon) := \sum_{j=1}^{N} a_j \phi(\varepsilon(x - x_j)) + \sum_{i=1}^{N_{m-1}} b_i p_i(x). \tag{4.6}$$

where $p_1, \dots, p_{N_{m-1}}$ is a basis for $\Pi_{<m}$. Due to [9, page 123], we find that

$$\sum_{j=1}^{N} a_j \phi(\varepsilon(x - x_j)) = \varepsilon^{-2K} \sum_{s=0}^{\infty} \varepsilon^{2(-q+s)} P_{-q+s}(x) \tag{4.7}$$

with

$$P_{-q+s}(x) = \sum_{\alpha \in I_{2s}} \left(\sum_{n=|\lfloor \frac{\alpha+\vec{1}}{2} \rfloor|_1}^{s} c_n \sum_{\beta \in I_{2n}^{\alpha}} (-1)^{|\alpha|} \frac{n!}{(\frac{\alpha+\beta}{2})!} \frac{(\alpha+\beta)!}{\alpha!\beta!} \sigma_{-q+s-n}^{[\beta]} \right) x^{\alpha}. \tag{4.8}$$

Applying Lemma 4.1, we note that all coefficients of x^α with $2s - m + 1 \le |\alpha| \le 2s$ becomes zero. Inserting (4.1) and (4.2) into (4.6), we obtain that

$$s(x, \varepsilon) = \varepsilon^{-2K} \sum_{s=0}^{\infty} \varepsilon^{2(-q+s)} (P_{-q+s}(x) + Q_{-q+s}(x)), \tag{4.9}$$

where

$$Q_{-q+s}(x) = \sum_{i=1}^{N_{m-1}} b_{i,-q+s} p_i(x). \tag{4.10}$$

For the sake of our argument, by cutting off the terms of degree α with $|\alpha| \le m - 1$ and absorbing them into the polynomial $Q_{-q+s}(x)$, we may set the polynomial $P_{-q+s}(x)$ as $P_{-q} = \cdots = P_{-q+m-1} = 0$ and

$$P_{-q+s}(x) = \sum_{\alpha \in I_{2s}, |\alpha|_1 \ge m} \left(\sum_{n=|\lfloor \frac{\alpha+\vec{1}}{2} \rfloor|_1}^{s} c_n \sum_{\beta \in I_{2n}^{\alpha}, |\beta|_1 \ge m} (-1)^{|\alpha|} \frac{n!}{(\frac{\alpha+\beta}{2})!} \frac{(\alpha+\beta)!}{\alpha!\beta!} \sigma_{-q+s-n}^{[\beta]} \right) x^{\alpha}. \tag{4.11}$$

Definition 4.2 *For $A_1 = \{1, \dots, p\}$ and the subsets $\{A_i\}_{i=1}^{\binom{d}{p}}$ of $\{1, \dots, d\}$ with p elements, let $I_{p,m,K}^i$ be the set of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $m \le |\alpha|_1 \le K$ and α_k for $k \in A_i$ are odd numbers, and the others are even numbers. There is a unique permutation between $I_{p,m,K}$ and $I_{p,m,K}^i$. The order of $I_{p,m,K}^i$ inherits from $I_{p,m,K}$ by the permutation.*

From the definition, we decompose the set $I_{m,K}$ of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ with $m \le |\alpha|_1 \le K$ into the disjoint union

$$I_{m,K} = \bigcup_{p,i} \left\{ I_{p,m,K}^i : p = 0, \dots, d \text{ and } i = 1, \dots, \binom{d}{p} \right\}.$$

In order to continue our argument, let us define the following two vectors

$$\begin{aligned} \mathbf{s}_{p,m,J}^i &:= (\sigma_{-q+n}^{[\beta]} : \beta = I_{p,m,J}^i(k),\ n = (J - |\beta|_1)/2), \\ \mathbf{p}_{p,m,J}^i &:= (P_{-q+s}[\alpha] : \alpha = I_{p,m,J}^i(k),\ s = (J + |\alpha|_1)/2), \end{aligned}$$

where $P_{-q+s}[\alpha]$ is the coefficient of x^α in $P_{-q+s}(x)$ (4.11). With the matrices from Definition 3.14 and the vectors defined above, we have a sequence of systems of equations for the discrete moments, **(Should $\mathbf{A}_{p,m,J}$ be changed to $\mathbf{A}_{p,m,J}^i$?)**

$$\mathbf{A}_{p,m,J} \mathbf{s}_{p,m,J}^i = \mathbf{p}_{p,m,J}^i, \quad i = 1, 2, \dots, \binom{d}{p} \quad \text{and} \quad J = m, m+1, \dots. \tag{4.12}$$

where p and J have the same parity and $0 \leq p \leq d$. By the assumption on ϕ and Theorem 3.9, the systems in (4.12) are nonsingular and we have a complete description of the relation between the discrete moments and the polynomials P_{-q+s}. From the coefficients of the polynomials, the system in (4.12) can be solved directly for determining the moments.

There is a range of ε-values for which we get a well-defined interpolant $s(x, \varepsilon)$ in (4.6) to the data. If we relate this to the expansion (4.9), we get the following conditions

- $P_K + Q_K$ interpolates the data at the points $X = \{x_j : j = 1, \ldots, N\}$ (4.13)
- $P_j + Q_j$ with $j \neq K$ interpolates 0 at the node points X.

Note that if $P_j + Q_j$ for $j \neq K$ is of degree $\leq m$, then the non-degeneracy property (3.12) of the node points on $\Pi_{<m}$ implies $P_j + Q_j = 0$. In particular, we have the following:

Lemma 4.3 *If $a_{j,-q} = 0$ for $j = 1, \ldots, N$, then $b_{j,-q} = 0$ for $j = 1, \ldots, N_{m-1}$.*

Proof. It is clear that $P_{-q}(x)$ in (4.8) is a zero polynomial. Since $P_{-q} + Q_{-q} = 0$, the non-degeneracy property (3.12) induces that the polynomial $Q_{-q} = 0$. The definition of Q_{-q} in (4.10) leads to the conclusion that $b_{j,-q} = 0$ with $j = 1, \ldots, N_m$. $\square$

Now, we are ready to prove Theorem 3.10. and Theorem 3.11 The technique for the proof of Theorem 3.10 is almost same as the method in [9, Theorem 4.2], hence is only sketch here.

Proof (a) of Theorem 3.10:

We consider the case when the point set X is of type (I), i.e., $\#X = N_{K,d}$ for some $K \geq m$ and the set X is unisolvent with respect to any basis for $\Pi_{<K+1}$. Also, the relation (4.4) holds for the basis $\{x^\alpha : \alpha \in I_K\}$. Due to the unisolvency of X, any polynomial of degree $\leq K$ that interpolate zero at N node points must be identically zero. In fact, applying the same analysis in [9, p 124] –where the only difference is that the linear system (4.12) is solved instead of $\mathbf{A}_{p,0,J}\mathbf{s}^i_{p,0,J} = \mathbf{p}^i_{p,0,J}$– to get

$$P_{-q+j} = 0, \quad j = m, \ldots, K-1, \quad \text{and} \quad \deg P_{-q+K} = K.$$

Here, let us consider the following two cases : (1) $P_{-q+K} = P_K$; (2) $P_{-q+K} \neq P_K$. Assume that $q > 0$ so that $P_{-q+K} \neq P_K$, which implies that $P_{-q+K} = 0$ since P_{-q+K} interpolates zero at the node points. Then we can solve the systems for $J = m, \ldots, K$, and we obtain that

$$\sigma^{[\beta]}_{-q} = 0, \quad |\beta|_1 = m, \ldots, K.$$

Combining this with Lemma 4.1 and (4.4), we obtain that $a_{j,-q} = 0$ for $j = 1, \ldots, N$. Further, Lemma 4.3 shows that $b_{j,-q} = 0$ for $j = 1, \ldots, N_m$, which is impossible since all of $a_j's$ and $b_i's$ are not a zero. Thus, we arrive at the conclusion that $q = 0$ and the interpolant $s(x, \varepsilon)$ in (4.9) becomes

$$s(x, \varepsilon) = P_K(x) + Q_K(x) + \varepsilon^2 \sum_{\ell=1}^{\infty} \varepsilon^{2(\ell-1)}(P_{K+\ell}(x) + Q_{K+\ell}(x)).$$

Unisolvency of X to $\Pi_{<K+1}$ ensures that $P_K + Q_K$ is the unique polynomial interpolant to the data on X. $\square$

In the case of type (II), the set X is unisolvent but the number of points does not coincide with the dimension of the polynomial space, i.e., $N_{K-1,d} < N < N_{K,d}$. Thus, the condition that a polynomial $p(x)$ of degree $\leq K$ interpolates zero at the node points does not ensure $p(x)$ to be identically zero any more, unless its degree is less than K.

Proof (b) of Theorem 3.10: The proof is almost same as the method in [9, Theorem 4.2], hence is only sketch here. First, the method in the proof of Theorem 3.10 induces that $P_{-q}+j=0$ for $j=m,\dots,K-1$ and $\deg P_K \leq K$. Second, using the same analysis in the proof of in [9, Theorem 4.2], we can show that P_{-q+K} is generated in terms of N linearly independent polynomials of degree $\leq K$. Then, we are in a position to get $P_{-q+K}=P_K$. Suppose that $q>0$. Then P_{-q+K} interpolates zero at the points so that $P_{-q+K}=0$. Applying (4.12) for $0\leq J\leq K$, we obtain that $\sigma_{-q}^{[\beta]}=0$ for $|\beta|_1\leq K$. It follows from (4.4) and Lemma 4.3 that $a_{1,-q}=\cdots=a_{N,-q}=b_{1,-q}=\cdots=b_{N_{m-1},-q}=0$, which is a contradiction. Thus, we must have $q=0$. Hence, as $\varepsilon\to 0$, the RBF interpolant converges to the polynomial P_K □

Proof (c) of Theorem 3.10:

In this case, in order that the limit exists as $\varepsilon\to\infty$, the interpolant $s(x,\varepsilon)$ in (4.9) must be expressed as

$$s(x,\varepsilon)=P_K(x)+Q_K(x)+\varepsilon^2(P_{K+1}(x)+Q_{K+1}(x))+\cdots.$$

Thus, the limit is, if exists, the polynomial $P_K(x)+Q_K(x)$ of degree $\leq 2K-m$. □

Next, we consider the case that ϕ be a conditionally positive definite function of order $m>0$ and the interpolant is from the space $\operatorname{span}\{\phi(\varepsilon(x-x_j)) : x_j\in X\}$.

Proof of Theorem 3.11:

Note that the RBF interpolant $s(\cdot,\varepsilon)$ is represented as

$$s(x,\varepsilon)=\sum_{j=1}^{N}a_j\phi(\varepsilon(x-x_j))=\varepsilon^{-2K}\sum_{s=0}^{\infty}\varepsilon^{2(-q+s)}(P_{-q+s}^{[m]}(x)+Q_{-q+s}^{[m]}(x))$$

with

$$P_{-q+s}^{[m]}(x)=\sum_{m\leq|\alpha|_1\leq 2s}P_{-q+s}[\alpha]x^\alpha,\quad\text{and}\quad Q_{-q+s}^{[m]}(x)=\sum_{|\alpha|_1<m}\tilde{P}_{-q+s}[\alpha]x^\alpha,$$

where $P_{-q+s}[\alpha]$ is the coefficient of x^α in $P_{-q+s}(x)$ (4.8). Applying the same argument as used in the proof of Theorem 3.10, we obtain the required results. □

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