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SPECTRAL APPROXIMATION ORDERS OF RADIAL BASIS FUNCTION INTERPOLATION ON THE SOBOLEV SPACE*

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Abstract. In this study, we are mainly interested in error estimates of interpolation, using smooth radial basis functions such as multiquadrics. The current theories of radial basis function interpolation provide optimal error bounds when the basis function ϕ is smooth and the approximand f is in a certain reproducing kernel Hilbert space $\mathcal{F}_\phi$. However, since the space $\mathcal{F}_\phi$ is very small when the function ϕ is smooth, the major concern of this paper is to prove approximation orders of interpolation to functions in the Sobolev space. For instance, when ϕ is a multiquadric, we will observe the error bound $o(h^k)$ if the function to be approximated is in the Sobolev space of smoothness order k.

Key words. radial basis function, interpolation, Sobolev space, positive definite function, multiquadric, "shifted" surface spline

AMS subject classifications. 41A05, 41A15, 41A25, 41A30, 41A63

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1. Introduction.

Radial basis function interpolation is a very useful and convenient tool for multivariate scattered data approximation problems. Its strengths are as follows: (i) it facilitates the evaluation of the approximant; (ii) the accuracy of approximation is usually very satisfactory provided the approximand f is reasonably smooth; (iii) there is enough flexibility in the choice of basis functions. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radial in the sense that $\phi(x) = \Phi(|x|)$, where $|\cdot|$ is the usual Euclidean norm.

Let Π_m denote the subspace of $C(\mathbb{R}^d)$ consisting of all algebraic polynomials of degree less than m on $\mathbb{R}^d$. Suppose that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is known only at a set of discrete points $X := \{x_1, \ldots, x_N\}$ in $\Omega \subset \mathbb{R}^d$. Radial basis function interpolation to f on X starts with choosing a basis function ϕ, and then it defines an interpolant by

$$a_{f,X}(x) := \sum_{i=1}^{\ell} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi(x - x_j), \tag{1.1}$$

where $p_1, \ldots, p_\ell$ is a basis for Π_m and the coefficients α_j $(j = 1, \ldots, N)$ and β_i $(i = 1, \ldots, \ell)$ are chosen to satisfy the linear system

$$\begin{aligned} a_{f,X}(x_j) &= f(x_j), \quad && j = 1, \ldots, N, \\ \sum_{j=1}^{N} \alpha_j p_i(x_j) &= 0, && i = 1, \ldots, \ell. \end{aligned} \tag{1.2}$$

Here, the set of scattered points X has the nondegeneracy property for Π_m; that is, if $p \in \Pi_m$ and $p(x_j) = 0$, $j = 1, \ldots, N$, then $p = 0$. It guarantees that the interpolation

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method reproduces the polynomial space Π_m, i.e., $a_{p,X} = p$ for any $p \in \Pi_m$. For a wide choice of functions ϕ and polynomial orders m, the existence and uniqueness of the solution of the linear system (1.2) is ensured when ϕ is a conditionally positive definite function (see [M]).

DEFINITION 1.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We say that ϕ is conditionally positive definite of order $m \in \mathbb{N} := \{1, 2, \ldots\}$ if for every finite set of pairwise distinct points $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ and for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N \setminus 0$ satisfying*

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \quad p \in \Pi_m,$$

the quadric form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i - x_j)$$

is positive definite.

In what follows, we assume $\phi = \Phi(|\cdot|)$ to be conditionally positive definite of order m. Also, the function ϕ is considered as a tempered distribution in $\mathcal{D}'(\mathbb{R}^d)$, and we assume that its Fourier transform $\hat{\phi}$ coincides on $\mathbb{R}^d \setminus 0$ with some continuous function while having a certain type of singularity (necessarily of a finite order) at the origin, i.e., $\hat{\phi}$ is of the form

$$|\cdot|^n \hat{\phi} = F > 0, \quad n \geq 0, \quad \text{and } F \in L_\infty(\mathbb{R}^d).$$

Among many radial basis functions, our major concern is with smooth functions ϕ such as multiquadrics $\phi(x) := c_{m,d}(|x|^2 + \lambda^2)^{m-d/2}$, d odd, $m > d/2$, where $c_{m,d}$ is a suitable constant.

For a given basis function ϕ, there arises a function space

$$\mathcal{F}_\phi := \left\{ f : |f|_\phi := \int_{\mathbb{R}^d} \frac{|\hat{f}(\theta)|^2}{\hat{\phi}(\theta)} d\theta < \infty \right\}, \tag{1.3}$$

which is called reproducing kernel Hilbert space (or "native" space) for ϕ ([MN2] and [WS]). For all $x \in \Omega$, $f \in \mathcal{F}_\phi$, bounds for the interpolation error are usually of the form

$$|f(x) - a_{f,X}(x)| \leq P_{\phi,X}(x)|f|_\phi. \tag{1.4}$$

Here $P_{\phi,X}$ is the *power function* that evaluates the norm of the error functional at x:

$$P_{\phi,X}(x) = \sup_{|f|_\phi \neq 0} \frac{|f(x) - a_{f,X}(x)|}{|f|_\phi}.$$

In fact, when the basis function ϕ is smooth, the interpolation method provides optimal asymptotic decay of errors, but the space $\mathcal{F}_\phi$ is very small. The approximands f need to be extremely smooth for effective error estimates. However, practically, most multivariate scattered data are not arising from extremely smooth functions. An error analysis for the case that the underlying function is reasonably smooth needs to be provided. Thus, the main objective of this paper is to prove asymptotic error bounds of interpolation (by using smooth basis function ϕ) to functions in a larger space, especially in the Sobolev space.

Asymptotic approximation properties are usually quantified by the notion of approximation order. In order to make this notion feasible, we measure the "density" of X (in Ω) by

$$h := h(X;\Omega) := \sup_{x\in\Omega} \min_{x_j\in X} |x - x_j|. \tag{1.5}$$

Here we assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain with both cone property and Lipschitz boundary. In particular, for a given set X, we adopt the scaled basis functions $\phi_\omega := \phi(\cdot/\omega)$, where

$$\omega := \omega(h)$$

is a parameter depending on h such that $h/\omega \to 0$ as $h \to 0$, and we use the notation

$$s_{f,X}(x) := \sum_{i=1}^{\ell} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi_\omega(x - x_j) \tag{1.6}$$

to differentiate from the notation $a_{f,X}$ in (1.1). Then our goal is to provide error estimates of $f - s_{f,X}$ of the following form: Let ϕ be a smooth basis function (e.g., multiquadric). Under some suitable conditions of the parameter ω (e.g., $\omega = h^r$ with $r \in [0, 1)$), we will show the asymptotic property

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k), \quad h \to 0,$$

provided that $f \in W^k_\infty(\Omega)$, the L_∞-Sobolev space of smoothness order k. To the writer's knowledge, this is the first paper dedicated to the study of spectral approximation order of interpolation to the functions in the Sobolev space $W^k_\infty(\Omega)$. Indeed, Buhmann and Dyn also explored the spectral convergence order of multiquadric interpolation in the paper [BuD]. However, this result considers interpolants on $h \cdot \mathbb{Z}^d$ under some conditions of the underlying function f, while we work with a finite subset X in Ω.

The reader who is interested in knowing more about the state of the art in the area of radial basis function methods may find it useful to consult with the surveys [Bu], [D], and [P]. Other important sources are the works of Wu and Schaback [WS] and especially those of Madych and Nelson [MN1], [MN2], who developed a theory of interpolation based on reproducing kernel Hilbert spaces. Interpolation by compactly supported basis functions has been studied by Wendland [W].

The following notations will be used throughout this paper. For any $k \in \mathbb{N}$, the Sobolev space is defined by

$$W^k_p(\Omega) := \left\{ f : \|f\|_{k,L_p(\Omega)} := \left(\sum_{|\alpha|_1 \le k} \|D^\alpha f\|^p_{L_p(\Omega)} \right)^{1/p} < \infty \right\}$$

with $1 \le p \le \infty$. Several different function norms are used. When $\mathbf{g}$ is a matrix or a vector, $\|\mathbf{g}\|_p$ indicates its p-norm with $1 \le p \le \infty$. For $x \in \mathbb{R}^d$, $|x| := (x_1^2 + \cdots + x_d^2)^{1/2}$ stands as its Euclidean norm. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is defined as

$$\hat{f}(\theta) := \int_{\mathbb{R}^d} f(t) \exp(-i\theta \cdot t)\, dt.$$

Also, for a function $f \in L_1(\mathbb{R}^d)$ we use the notation $f^\vee$ for the inverse Fourier transform. In particular, the Fourier transform can be uniquely extended to the space of tempered distributions on $\mathbb{R}^d$.

2. The extension of a function f in $W_\infty^k(\Omega)$. Our analysis in this paper requires the construction of a suitable extension of a given function $f \in W_\infty^k(\Omega)$ to a function on $\mathbb{R}^d$. Indeed, the lengthy assumptions on Ω in section 1 assure the existence of a function on $\mathbb{R}^d$ whose restriction to Ω agrees with f. The following result is cited from literature.

THEOREM 2.1 (Brenner and Scott [BrS]). *Suppose that Ω has a Lipschitz boundary. Then for every function $f \in W_p^k(\Omega)$, there is an extension mapping $E : W_p^k(\Omega) \to W_p^k(\mathbb{R}^d)$ defined for all nonnegative integer k and real numbers p in the range $1 \le p \le \infty$ satisfying $Ef|_\Omega = f$ for all $f \in W_p^k(\Omega)$ and*

$$\|Ef\|_{k,L_p(\mathbb{R}^d)} \le c\|f\|_{k,L_p(\Omega)},$$

where the constant c is independent of f.

The construction of our suitable extension of $f \in W_\infty^k(\Omega)$ to a function on $\mathbb{R}^d$ can be done in two steps. First, according to Theorem 2.1, there exists a function $Ef \in W_\infty^k(\mathbb{R}^d)$ such that $Ef|_\Omega = f$. Second, we let σ_Ω be a C^∞-cutoff function such that $\sigma_\Omega(x) = 1$ for $x \in \Omega$ and $\sigma_\Omega(x) = 0$ for $|x| > r$ with a sufficiently large $r > 0$. Then we define an extension f^o by

$$f^o := \sigma_\Omega Ef.$$

Of course, f^o is compactly supported and $f^o(x) = f(x)$ for $x \in \Omega$. Indeed, for a large part of this paper, we wish to work with f^o and not f. For convenience, we will henceforth write f for f^o. Therefore, here and in what follows, *without great loss*, we assume that an approximand $f \in W_\infty^k(\Omega)$ is supported in a sufficiently large compact set in $\mathbb{R}^d$ such that $f \in W_\infty^k(\mathbb{R}^d)$.

3. Error bounds. In this section, we will provide a (modified) method of error analysis of interpolation to functions in the Sobolev space $W_\infty^k(\Omega)$. In addition, we obtain a sufficient condition for the optimal convergence order $\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k)$ with $f \in W_\infty^k(\Omega)$. For this purpose, we start with finding a mollified function (say, f_H) of a given (underlying) function f. The function f_H is supposed to be in the space $\mathcal{F}_{\phi_\omega}$ in (1.3) and should be a good approximation to f in some sense. In order to define a mollification f_H of f, we use a nonnegative C^∞-cutoff function

$$\sigma : \mathbb{R}^d \to [0, 1]. \tag{3.1}$$

Here, for convenience, we assume that the function σ is radially symmetric and supp σ lies in the Euclidean ball $B_1 = \{x \in \mathbb{R}^d : |x| \le 1\}$, and we assume that $\sigma = 1$ on $B_{1/2}$ and $\|\sigma\|_{L_\infty(\mathbb{R}^d)} = 1$. Then, letting $\sigma_\delta := \sigma(\cdot/\delta)$ with $\delta > 0$, we construct two functions f_H and f_T by

$$\begin{aligned} f_H &:= \sigma_\delta(h\cdot)^\vee * f, \\ f_T &:= f - \sigma_\delta(h\cdot)^\vee * f. \end{aligned} \tag{3.2}$$

It clearly follows that

$$f = f_H + f_T.$$

Also, due to the fact that the interpolation operator $s_{f,X}$ is linear, it is useful to split the error $f - s_{f,X}$ as follows:

$$f - s_{f,X} = (f_H - s_{f_H,X}) + (f_T - s_{f_T,X}).$$

Accordingly, this section falls naturally into two parts. In the first, since $f_H \in \mathcal{F}_{\phi_\omega}$, we estimate the term $f_H - s_{f_H,X}$ by applying the well-known method in (1.4). The second part of the section deals with $f_T - s_{f_T,X}$. Our main tool for this case is to use stability results on the interpolation process. Afterward, the final result is stated in Theorem 3.6

From the papers (see, e.g., [WS], [MN2]), we cite the following lemma.

LEMMA 3.1. *Let $a_{X,f}$ in* (1.1) *be an interpolant to f on $X = \{x_1, \ldots, x_N\}$. Given ϕ and m, for all functions f in the space $\mathcal{F}_\phi$, there is an error bound of the form*

$$|f(x) - a_{f,X}(x)| \le |f|_\phi P_{\phi,X}(x),$$

where $P_{\phi,X}(x)$ is the norm of the error functional at x, i.e.,

$$P_{\phi,X}(x) = \sup_{|f|_\phi \neq 0} \frac{|f(x) - a_{f,X}(x)|}{|f|_\phi}. \tag{3.3}$$

The following lemma estimates the error $f_H - s_{f_H,X}$.

LEMMA 3.2. *Let $f_H := \sigma_\delta(h\cdot)^\vee * f$ with $\sigma_\delta(h\cdot)$ as the cutoff function in* (3.1), *and let $s_{f_H,X}$ in* (1.6) *be the interpolant to f_H on X using the basis function ϕ_ω. Let ω be a parameter depending on h, i.e., $\omega = \omega(h)$. Then, for every $f \in L_2(\mathbb{R}^d)$, we have an estimate of the form*

$$|f_H(x) - s_{f_H,X}(x)| \le P_{\phi,X/\omega}(x/\omega) M_{\phi,\omega}(\delta/h) \|f\|_{L_2(\mathbb{R}^d)}, \quad x \in \Omega,$$

where $M_{\phi,\omega}(r)$, $r > 0$, is defined by

$$M_{\phi,\omega}(r) := \sup_{\theta \in B_r} |\hat{\phi}_\omega(\theta)|^{-1/2}. \tag{3.4}$$

Proof. Recalling the definition of $s_{f_H,X}$ in (1.6), one simply notes that the function $s_{f_H,X}(\omega\cdot)$ can be considered as an interpolant (employing the shifts of ϕ) to the scaled function $f_H(\omega\cdot)$ on X/ω, i.e.,

$$s_{f_H,X}(\omega\cdot) = \sum_{i=1}^{\ell} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi(\cdot - x_j/\omega) = a_{f_H(\omega\cdot),X/\omega},$$

with $a_{f,X}$ in (1.1). Then, since $f_H(\omega\cdot)$ belongs to the space $\mathcal{F}_\phi$, Lemma 3.1 can be used directly to derive the bound

$$\begin{aligned} |f_H(x) - s_{f_H,X}(x)| &= |f_H(\omega\cdot) - a_{f_H(\omega\cdot),X/\omega}|(x/\omega) \\ &\le P_{\phi,X/\omega}(x/\omega)|f_H(\omega\cdot)|_\phi. \end{aligned} \tag{3.5}$$

Now, in order to estimate the term $|f_H(\omega\cdot)|_\phi$, we find from the definition of f_H in (3.2) that

$$\widehat{f_H(\omega\cdot)}(\theta) = \omega^{-d}\sigma_\delta(h\theta/\omega)\hat{f}(\theta/\omega).$$

Then the explicit formula of the norm $|\cdot|_\phi$ in (1.3) induces by change of variables that

$$|f_H(\omega\cdot)|_\phi^2 = \omega^{-d}\int_{\mathbb{R}^d} \left|\sigma_\delta(h\theta)\hat{f}(\theta)\right|^2 \hat{\phi}^{-1}(\omega\theta)d\theta$$
$$\le \sup_{\theta\in B_{\delta/h}} |\hat{\phi}_\omega(\theta)|^{-1}\|f\|^2_{L_2(\mathbb{R}^d)}.$$

Due to the expression (3.5), we finish the proof. □

Now, we are going to turn to the estimate of the error $f_T - s_{f_T,X}$ with f_T in (3.2). Since there is no guarantee that the function f_T belongs to the space $\mathcal{F}_\phi$, the classical method of the error analysis of interpolation is not applicable to this case. Hence, in order to make the estimate $f_T - s_{f_T,X}$ feasible, we employ the stability results on interpolation process. To this end, we define the separation distance within X by

$$q := q_X := \min_{1\le i\ne j\le N} |x_i - x_j|/2. \tag{3.6}$$

It is well known from literature (e.g., [NSW2], [S1]) that as q is getting smaller, the condition number of the interpolation matrix becomes larger. Also, the irregularity of a set X can be measured by the ratio h/q. In particular, we assume that the sets of scattered points considered in this study are sets of quasi-uniformly distributed points. These sets satisfy the following property: There exists a constant $\eta > 0$ independent of X such that

$$2q \le h \le \eta q. \tag{3.7}$$

This condition implies that the number of the scattered points in the set X is bounded above by a quantity that depends on the density of X, i.e., $N = O(h^{-d})$. On the other hand, we particularly introduce a function φ defined by

$$\varphi := \sigma_\epsilon^\vee = \sigma(\cdot/\epsilon)^\vee, \tag{3.8}$$

where σ_ϵ is the cutoff function in (3.1). For the purpose of simplifying the following analysis, we assume $\epsilon > 0$ to be any fixed number satisfying the condition

$$\epsilon < \delta/\eta \tag{3.9}$$

with δ in (3.2). It is obvious that the Fourier transform of φ is $\hat{\varphi} = \sigma_\epsilon$, which is supported in the ball B_ϵ. Furthermore, since σ is a C^∞-cutoff function, $\varphi(x)$ decays fast as x tends to ∞. Indeed, the function φ is employed to use the stability results on the interpolation process. It first requires us to show that φ is a conditionally positive definite radial function. For this proof, we find the following identity:

$$\sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\varphi(x_i - x_j) = \int_{\mathbb{R}^d} \hat{\varphi}(\theta)\left|\sum_{j=1}^N \alpha_j e^{ix_j\cdot\theta}\right|^2 d\theta$$

for any $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N \setminus 0$. Since the map $\theta \mapsto \sum_{j=1}^N \alpha_j e^{ix_j\cdot\theta}$, $\theta\in\mathbb{R}^d$, has zeros at most on a set of measure zero, we see that the integral in the right-hand side of the above identity is always positive. It is asserted from Definition 1.1 that the function φ is conditionally positive definite of order $m = 0$. Also, since the cutoff function σ_ϵ is radially symmetric, its inverse Fourier transform φ is also a radial function (see [S3]). Then an interpolant to f on X using the (scaled) function

$$\varphi_q(x) := \varphi(x/q)$$

is of the form

$$g_{f,X}(x) = \sum_{j=1}^{N} \beta_j \varphi_q(x - x_j). \tag{3.10}$$

One simply notes that the matrix $\mathbf{A}_{\varphi_q} := (\varphi_q(x_i - x_j))_{i,j=1,\ldots,N}$ is positive definite.

PROPOSITION 3.3. *Let X be a q-separated set with q in equation* (3.6)*. Let $\mathbf{b}_f := (\beta_1, \ldots, \beta_N)^T$, and let $\mathbf{A}_{\varphi_q} := (\varphi_q(x_i - x_j))_{i,j=1,\ldots,N}$ be the interpolation matrix by φ_q. Then we have the following properties:*

(a) $\|\mathbf{A}_{\varphi_q}^{-1}\|_2 \le c_1$ *for some* $c_1 > 0$.

(b) $\|\mathbf{A}_{\varphi_q}^{-1}\|_1 = \|\mathbf{A}_{\varphi_q}^{-1}\|_\infty \le c_2 \|\mathbf{A}_{\varphi_q}^{-1}\|_2$ *for some* $c_2 > 0$.

(c) $\|\mathbf{b}_f\|_\infty \le c_3 \|f\|_{L_\infty(\mathbb{R}^d)}$ *for some* $c_3 > 0$.

Proof. Since the interpolation matrix $\mathbf{A}_{\varphi_q}$ has the separation distance 1, the matrix norm $\|\mathbf{A}_{\varphi_q}^{-1}\|_2$ is bounded by a constant (see [NSW2]). Furthermore, the basis function φ_q decays fast around ∞, and the inequality in (b) is proved by a direct application of Theorem 3.11 in the paper [BSW]. The identity $\|\mathbf{A}_{\varphi_q}^{-1}\|_1 = \|\mathbf{A}_{\varphi_q}^{-1}\|_\infty$ is an obvious consequence of symmetry. Finally, to prove (c), we find that the matrix $\mathbf{b}_f$ can be written as

$$\mathbf{b}_f = \mathbf{A}_{\varphi_q}^{-1}\mathbf{f}$$

with $\mathbf{f} := (f(x_1), \ldots, f(x_N))^T$. After some direct calculations, one can prove the inequality $\|\mathbf{b}_f\|_\infty \le \|\mathbf{A}_{\varphi_q}^{-1}\|_1 \|f\|_{L_\infty(\mathbb{R}^d)}$. Hence, by using (b), the relation in (c) is immediate. $\square$

Before estimating the error $f_T - s_{f_T,X}$, we cite the following result.

LEMMA 3.4 (Yoon [Y1]). *Let $f_T = f - \sigma_\delta(h\cdot)^\vee * f$ with $\sigma_\delta(h\cdot)$ the cutoff function in* (3.1)*. Then, for every $f \in W_\infty^k(\mathbb{R}^d)$ with k a positive integer, we have the following decaying property:*

$$\|f_T\|_{L_\infty(\mathbb{R}^d)} = \|f - f_H\|_{L_\infty(\mathbb{R}^d)} = o(h^k).$$

LEMMA 3.5. *Let X be a set of scattered points with the condition* (3.7)*, and let $s_{f_T,X}$ in* (1.6) *be the interpolant to f_T on X using ϕ_ω, where $f_T = f - \sigma_\delta(h\cdot)^\vee * f$ and $\omega = \omega(h)$. Then, for every $f \in W_\infty^k(\Omega)$ with k a positive integer, there is an error bound of the form*

$$|f_T(x) - s_{f_T,X}(x)| \le o(h^k)(1 + P_{\phi,X/\omega}(x/\omega) M_{\phi,\omega}(\delta/h)), \quad x \in \Omega,$$

as $h \to 0$, with $M_{\phi,\omega}(r)$, $r > 0$, in (3.4).

Proof. Let us first define a function $\tilde{f}$ by

$$\tilde{f} := h^{-k} f_T.$$

It is clear that $h^{-k} s_{f_T,X} = s_{\tilde{f},X}$. Then, we employ the interpolant $g_{\tilde{f},X}$ in (3.10) to derive the following bound:

$$h^{-k}|f_T(x) - s_{f_T,X}(x)| \le |\tilde{f}(x)| + |g_{\tilde{f},X}(x)| + |g_{\tilde{f},X}(x) - s_{\tilde{f},X}(x)|.$$

The convergence property $\|f_T\|_{L_\infty(\mathbb{R}^d)} = o(h^k)$ in Lemma 3.4 yields that $\|\tilde{f}\|_{L_\infty(\mathbb{R}^d)} = o(1)$ as h tends to 0. Also, by applying Proposition 3.3, we get

$$\begin{aligned} |g_{\tilde{f},X}(x)| &\le \|\mathbf{b}_{\tilde{f}}\|_\infty \sum_{j=1}^N \varphi_q(x-x_j) \\ &\le c\,\|\tilde{f}\|_{L_\infty(\mathbb{R}^d)} = o(1). \end{aligned}$$

Here, since X is a q-separated set and the function φ_q decays fast around ∞, we can easily check that $\sum_{j=1}^N \varphi_q(\cdot - x_j)$ is uniformly bounded on Ω. Therefore, it remains to show that the term $g_{\tilde{f},X} - s_{\tilde{f},X}$ is bounded by $o(1)P_{\phi,X/\omega}(x/\omega)M_{\phi,\omega}(\delta/h)$ as $h \to 0$. For this, we claim that

$$s_{\tilde{f},X} = s_{g_{\tilde{f},X},X}.$$

In fact, this identity is immediate from the interpolation property $\tilde{f}(x_j) = g_{\tilde{f},X}(x_j)$ for any $j = 1, \dots, N$. Then, applying the same technique as in the proof of Lemma 3.2 gives us the bound

$$|s_{\tilde{f},X}(x) - g_{\tilde{f},X}(x)| \le P_{\phi,X/\omega}(x/\omega)|g_{\tilde{f},X}(\omega\cdot)|_\phi, \quad x \in \Omega. \tag{3.11}$$

Moreover, according to the definition of the norm $|\cdot|_\phi$, we get

$$\begin{aligned} |g_{\tilde{f},X}(\omega\cdot)|_\phi^2 &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \beta_j e^{ix_j\cdot\theta} \right|^2 \sigma_\epsilon^2(q\theta)\hat{\phi}_\omega^{-1}(\theta)q^{2d}d\theta \\ &\le M_{\phi,\omega}^2(\epsilon/q) \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \beta_j \varphi_q(x-x_j) \right|^2 dx. \end{aligned} \tag{3.12}$$

Remembering the relations $\frac{1}{q} \le \frac{\eta}{h}$ in (3.7) and $\epsilon < \frac{\delta}{\eta}$ in (3.9), we easily find that $\frac{\epsilon}{q} \le \frac{\eta\epsilon}{h} \le \frac{\delta}{h}$. Then since $M_{\phi,\omega}(r)$ is monotonically increasing as r grows, it follows that

$$M_{\phi,\omega}(\epsilon/q) \le M_{\phi,\omega}(\delta/h). \tag{3.13}$$

Also, since $\sum_{j=1}^N \varphi_q(\cdot - x_j)$ is uniformly bounded, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \beta_j \varphi_q(x-x_j) \right|^2 dx &\le c\|\mathbf{b}_{\tilde{f}}\|_\infty^2 \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \varphi_q(x-x_j) \right| dx \\ &\le c'\|\tilde{f}\|_{L_\infty(\mathbb{R}^d)} = o(1) \end{aligned} \tag{3.14}$$

by Proposition 3.3 and the condition $N = O(h^{-d})$. Hence, inserting (3.13) and (3.14) into (3.12), we arrive at the bound

$$|g_{f,X}|_\phi \le M_{\phi,\omega}(\delta/h)o(1).$$

Together with (3.11), we complete the proof of this lemma. ∎

From Lemma 3.2 and Lemma 3.5, we realize that one of the important ingredients for the estimate $f - s_{f,X}$ is the term $M_{\phi,\omega}(\delta/h)$, $\delta > 0$. Observing the definition of

f_H in (3.2) carefully, we find that the number δ can be chosen arbitrarily. Of course, a certain choice of δ should induce a suitable bound of $P_{\phi,X/\omega}(x/\omega)M_{\phi,\omega}(\delta/h)$, which leads to a desirable estimate of $f - s_{f,X}$. We are now ready to describe the main result of this section.

THEOREM 3.6. *Let X be a set of scattered points with the condition* (3.7)*, and let $s_{f,X}$ in* (1.6) *be an interpolant to f on X using the basis function $\phi_\omega = \phi(\cdot/\omega)$. Let $M_{\phi,\omega}(r)$, $r > 0$, be defined as in* (3.4)*. Assume that there exists a constant $\delta_0 > 0$ such that*

$$P_{\phi,X/\omega}(x/\omega)M_{\phi,\omega}(\delta_0/h) \leq o(h^k).$$

Then, for every function $f \in W^k_\infty(\Omega)$ with k a positive integer, we have an error bound of the form

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k).$$

4. Applications to special radial basis functions. We now turn to applications to special radial basis functions. Employing some known basis functions ϕ, we will show that the interpolant $s_{f,X}$ provides optimal approximation orders for $f \in W^k_\infty(\Omega)$. All the examples here are based on Theorem 3.6.

Example 4.1. Let the radial basis function ϕ be chosen to be one of the following:

(a) $\phi_\lambda(x) := (-1)^{\lceil m-d/2\rceil}(|x|^2 + \lambda^2)^{m-d/2}$, d odd, $m > d/2$ (multiquadrics),

(b) $\phi_\lambda(x) := (-1)^{m-d/2+1}(|x|^2 + \lambda^2)^{m-d/2}\log(|x|^2 + \lambda^2)^{1/2}$, $m > d/2$, d even ("shifted" surface splines).

(c) $\phi_\lambda(x) := (|x|^2 + \lambda^2)^{m-d/2}$, $0 < m < d/2$ (inverse multiquadrics),

where d, $m \in \mathbb{N}$ and $\lambda > 0$, and where $\lceil s\rceil$ indicates the smallest integer greater than s. Note that we stress the parameter λ by using the notation ϕ_λ. We find (see [GS]) that the Fourier transform of ϕ_λ is of the form

$$\hat{\phi}_\lambda = c(m,d)\tilde{K}_m(\lambda\cdot)|\cdot|^{-2m},$$

where $c(m,d)$ is a positive constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|) \neq 0$, $t \geq 0$, with $K_\nu(|t|)$ the modified Bessel function of order ν. It is well known from literature (e.g., [AS]) that

$$\tilde{K}_\nu \sim (1 + |\cdot|^{(2\nu-1)/2})\exp(-|\cdot|).$$

Then, for all $\theta \in B_{\delta/h}$, we have the bound $\hat{\phi}_\lambda(\omega\theta)^{-1/2} \leq c|\omega\delta/h|^m \exp(\lambda\omega\delta/2h)$ for a constant $c > 0$. It leads to the inequality

$$M_{\phi_\lambda,\omega}(\delta/h) \leq c(\delta)\omega^{-d/2}|\omega/h|^m \exp(\lambda\omega\delta/2h),$$

where $c(\delta)$ is a constant depending on δ. On the other hand, due to Madych and Nelson [MN3], we see that there exists a constant $c' > 0$ independent of X such that

$$P_{\phi_\lambda,X/\omega}(x/\omega) \leq c\exp(-c'\lambda\omega/h)$$

for a sufficiently small $h > 0$. Since $\omega^{m-d/2} \leq o(h^{-d/2})$ (see section 1), from the above two inequalities, we arrive at the expression

$$M_{\phi_\lambda,\omega}(\delta/h)P_{\phi_\lambda,X/\omega}(x/\omega) \leq c(\delta)h^{-m-d/2}\exp\left(-\frac{\lambda\omega}{h}(c' - \delta/2)\right). \tag{4.1}$$

Here, we can choose a sufficiently small $\delta_0 > 0$ such that $c' - \delta/2 > 0$ for any $\delta \leq \delta_0$. In particular, we assume ω to satisfy the relation

$$h|\log h|^{1+r} \leq \omega$$

for any fixed $r > 0$. Then, it follows that

$$\exp\left(-\frac{\lambda\omega}{h}(c' - \delta_0/2)\right) \leq \exp\big(-\lambda|\log h|^{1+r}(c' - \delta_0/2)\big) = h^{\lambda(c'-\delta_0/2)|\log h|^r}. \tag{4.2}$$

Indeed, as h tends to 0, the number $\lambda(c' - \delta_0/2)|\log h|^r > 0$ becomes arbitrarily large. Hence, for any given $k \in \mathbb{N}$, there exists a sufficiently small $h_0 > 0$ such that $h^{\lambda(c'-\delta/2)|\log h|^r} \leq o(h^{k+m+d/2})$ for any $h \leq h_0$. Consequently, together with (4.1) and (4.2), we conclude that

$$M_{\phi_\lambda,\omega}(\delta_0/h)P_{\phi_\lambda,X/\omega}(x/\omega) \leq o(h^k), \quad h \leq h_0.$$

According to Theorem 3.6, we have the following result.

THEOREM 4.1. *Let ϕ_λ be one of the radial basis functions: multiquadrics, inverse multiquadrics, and "shifted" surface splines. Let X be a set of scattered points with the condition* (3.7), *and let $s_{f,X}$ in* (1.6) *be an interpolant to f on X using $\phi_\lambda(\cdot/\omega)$. Assume that $\omega = \omega(h)$ is chosen to satisfy the relation*

$$h|\log h|^{1+r} \leq \omega$$

for any fixed $r > 0$. Then, for every $f \in W^k_\infty(\Omega)$ with k a positive integer, we have an error estimate of the form

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k) \quad \textit{as} \quad h \to 0.$$

COROLLARY 4.2. *Let ϕ_λ be one of the radial basis functions: multiquadrics, inverse multiquadrics, and "shifted" surface splines. Let X be a set of scattered points with the condition* (3.7), *and let $s_{f,X}$ in* (1.6) *be an interpolant to f on X using $\phi_\lambda(\cdot/\omega)$. Assume that $\omega(h) = h^s$ with $s \in [0, 1)$ or $\omega(h) = h|\log h|^{1+r}$ with $r > 0$. Then, for every $f \in W^k_\infty(\Omega)$ with k a positive integer,*

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k) \quad \textit{as} \quad h \to 0.$$

Remark. Recalling that the interpolant $a_{f,X}$ in (1.1) uses the original (nonscaled) basis function, we make an observation concerning the interpolants $a_{f,X}$ in relation to $s_{f,X}$. Given a set X, assume that the interpolant $a_{f,X}$ employs the basis function $\phi_{\omega\lambda}$ instead of ϕ_λ. Then, one should realize that the interpolant $a_{f,X}$ is identically equal to $s_{f,X}$, which uses ϕ_λ. The equality can be verified by the uniqueness of the solution of the linear system (1.2). The reader is referred to the paper [Y2] for the details of the proof.

Example 4.2. Let us consider the basis function ϕ whose Fourier transform $\hat{\phi}$ is of the form

$$\hat{\phi}(\theta) = \exp(-|\theta|^a)$$

with $0 < a \leq 1$. In the case $a = 1$, the basis function ϕ becomes the so-called Poisson kernel

$$\phi = \frac{c_d}{(1 + |\cdot|^2)^{(d+1)/2}}$$

with a suitable constant c_d. For any $\theta \in B_{\delta/h}$, we get $\hat{\phi}(\omega\theta)^{-1} \le \exp((\omega\delta/h)^a)$. It leads to the inequality

$$M_{\phi,\omega}(\delta/h) \le \omega^{-d/2} \exp((\omega\delta)^a/h^a).$$

Also, due to Madych and Nelson (see [MN3]), there exists a constant $c' > 0$ independent of X such that

$$P_{\phi,X/\omega}(x/\omega) \le c \exp(-c'\omega^a/h^a)$$

for sufficiently small $h > 0$. Invoking the condition $\omega^{-d/2} \le o(h^{-d/2})$, we derive from the above inequalities that

$$M_{\phi,\omega}(\delta/h) P_{\phi,X/\omega}(x/\omega) \le c h^{-d/2} \exp\left(-\frac{\omega^a}{h^a}(c' - \delta^a)\right). \tag{4.3}$$

Now, in a similar fashion to the case of Example 4.1, we can choose a sufficiently small $\delta_0 > 0$ such that $c' - \delta^a > 0$ for any $\delta \le \delta_0$. In particular, we assume ω to satisfy

$$h^a |\log h|^{1+r} \le \omega^a$$

for any fixed $r > 0$. Then, it follows that

$$\begin{aligned}\exp\left(-\frac{\omega^a}{h^a}(c' - \delta_0^a)\right) &\le \exp\big(-|\log h|^{1+r}(c' - \delta_0^a)\big)\\ &= h^{|\log h|^r (c' - \delta_0^a)}.\end{aligned}$$

Here, $|\log h|^r(c' - \delta_0^a)$ becomes arbitrarily large as h tends to 0. Thus, for any given $k \in \mathbb{N}$, there exists a sufficiently small $h_0 > 0$ such that $h^{|\log h|^r(c'-\delta^a)} \le o(h^{k+d/2})$ if $h \le h_0$. Therefore, together with (4.3), we conclude that

$$M_{\phi_\lambda,\omega}(\delta/h) P_{\phi_\lambda,X/\omega}(x/\omega) \le o(h^k), \quad h \le h_0.$$

According to Theorem 3.6, we have the following result.

THEOREM 4.3. *Let ϕ be the basis function whose Fourier transform $\hat{\phi}$ is defined by $\hat{\phi} = \exp(-|x|^a)$ with $a \le 1$. Let X be a set of scattered points with the condition (3.7), and let $s_{f,X}$ in (1.6) be an interpolant to f on X using $\phi_\lambda(\cdot/\omega)$. Assume that $\omega = \omega(h)$ is chosen to satisfy the relation*

$$h|\log h|^{(1+r)/a} \le \omega$$

with a fixed number $r > 0$. Then, for every $f \in W_\infty^k(\Omega)$ with k a positive integer, we have an error bound of the form

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k) \quad \text{as} \quad h \to 0.$$

COROLLARY 4.4. *Let ϕ be the function whose Fourier transform $\hat{\phi}$ is of the form $\hat{\phi} = \exp(-|x|^a)$ with $0 < a \le 1$. Let X be a set of scattered points with the condition (3.7), and let $s_{f,X}$ in (1.6) be an interpolant to f on X using $\phi_\lambda(\cdot/\omega)$. Assume that $\omega = h^s$ with $s \in [0, 1)$ or $\omega = h|\log h|^{(1+r)/a}$ with $r > 0$. Then, for every $f \in W_\infty^k(\Omega)$ with k a positive integer,*

$$\|f - s_{f,X}\|_{L_\infty(\Omega)} = o(h^k) \quad \text{as} \quad h \to 0.$$

Remark. The Gaussian function $\phi := \exp(-\alpha|\cdot|^2)$, $\alpha > 0$, is not included in the examples of section 4. Indeed, the "quadratic exponential" error bound $\exp(-c/h^2)$, $c > 0$, of its power function $P_{\phi,X}$ is necessary to obtain the condition

$$P_{\phi,X/\omega}(x/\omega)M_{\phi,\omega}(\delta_0/h) \leq o(h^k)$$

for some $\delta_0 > 0$. However, it is not yet proven in the bounded domain case, but it is shown only on all of $\mathbb{R}^d$ under certain circumstances. The reader is referred to the manuscript [S3] for the details. More generally, for any given basis function ϕ, there would be a general theorem on the bounds of $P_{\phi,X}$ in terms of $\hat{\phi}$. In fact, we can easily check that $P_{\phi,X}$ is dependent only on the Fourier transform $\hat{\phi}$, more precisely, on the decaying property of $\hat{\phi}$ (see [WS] and [MN2] for the details).

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