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Stationary subdivision schemes reproducing polynomials

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Abstract

A new class of subdivision schemes is presented. Each scheme in this class is a quasi-interpolatory scheme with a tension parameter, which reproduces polynomials up to a certain degree. We find that these schemes extend and unify not only the well-known Deslauriers–Dubuc interpolatory schemes but the quadratic and cubic B-spline schemes. This paper analyzes their convergence, smoothness and accuracy. It is proved that the proposed schemes provide at least the same or better smoothness and accuracy than the aforementioned schemes, when all the schemes are based on the same polynomial space. We also observe with some numerical examples that, by choosing an appropriate tension parameter, our new scheme can remove undesirable artifacts which usually appear in interpolatory schemes with irregularly distributed control points.

Keywords: Subdivision scheme; Quasi-interpolation; Polynomial reproduction; Smoothness; Approximation order

1. Introduction

In recent decades, subdivision schemes have become important and efficient ways to generate smooth curves and surfaces. Given a set of control points at level 0, a subdivision rule is applied iteratively to generate a new (denser) set of control points. If the refinement rule is the same at all levels and positions of iteration, the scheme is called a *stationary subdivision scheme*. Under a suitable condition on the subdivision rule, the sequence of control points converges to a smooth limit curve or surface. The convergence analysis of stationary subdivision schemes can be found in (Cavaretta et al., 1991; Dyn, 1992; Deslauriers and Dubuc, 1989; Dyn et al., 1987; Dyn and Levin, 1991). The most familiar examples of such schemes are subdivision of B-splines (Cohen et al., 1980) and 4-(or 6-)point interpolatory scheme (Dyn et al., 1987), and polynomial-based interpolatory schemes studied independently by Deslauriers and Dubuc (1989). Recently, a new four-point subdivision scheme that generates C^2 curves has been introduced by Dyn et al. (2004).

In this paper, we consider subdivision schemes for curves. Since each component of the curve is a scalar function generated by the same subdivision scheme, the analysis of a subdivision scheme can be reduced to the case of initial

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control points in $\mathbb{R}$. The refinement rule is represented by a subdivision operator S, and it defines new sets of points $f^k = \{f_n^k\colon n \in \mathbb{Z}\}$ at each level $k \geqslant 0$ from a given set of control points at level zero $f^0 = \{f_n^0\colon n \in \mathbb{Z}\}$ formally by

$$f^k = S^k f^0.$$

A subdivision scheme is termed *uniformly convergent*, if for every initial data $f^0 = \{f_n^0 \in \mathbb{R}\colon n \in \mathbb{Z}\} \in \ell^\infty(\mathbb{Z})$, there exists a function $f \in C^0(\mathbb{R})$ such that, for any interval $[a, b]$,

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}\cap 2^k[a,b]} \left|f_n^k - f(2^{-k}n)\right| = 0, \tag{1}$$

and $f \not\equiv 0$ for some initial data f^0. The function f in (1) is denoted by $S^\infty f^0$, and called a limit function of S. In particular, the *basic limit function* of S is defined by

$$\varphi = S^\infty \delta, \tag{2}$$

with the initial data $\delta = \{\delta_{0,n}\colon n \in \mathbb{Z}\}$, where $\delta_{0,n}$ denotes the Kronecker delta. The scheme S is called C^γ, if its basic limit function φ (and hence all functions generated by it) is C^γ.

Among the many common criteria for a convergent subdivision scheme S, two most important ones are the smoothness and the approximation power of the scheme S. Indeed, the approximation power of a subdivision scheme can be quantified by the polynomial reproducing property. A stationary subdivision scheme S reproducing polynomials is called a *quasi-interpolatory* scheme.

It is well known that the B-spline subdivision schemes provide the optimal smoothness with the minimal support of the basic limit function φ in (2) (Dyn, 1992). However, it reproduces only linear polynomials unless some suitable operator is applied to each f^k (Levin, 2003). On the other hand, the Deslauriers–Dubuc schemes provide the optimal approximation order. But they are less smooth than B-spline schemes, when they are based on the same polynomial space. For instance, the 4-point Deslauriers–Dubuc scheme generates C^1 curves while the cubic B-spline is C^2. Further, interpolation, in spite of being a very desirable property in curve (and surface) design, may raise twisting artifacts to the parametric curves if the initial control points are irregular.

In view of the above discussion, there is a need for subdivision schemes that combine the advantages of the aforementioned schemes, while overcoming their drawbacks at the same time. To this end, we propose in this paper a new class of subdivision schemes that unifies the Deslauriers–Dubuc schemes and the quadratic and cubic B-spline schemes. However, higher order B-splines are not included in this class because our primary concern is to construct subdivision schemes with high approximation orders. Specifically, the contribution of our schemes can be written as follows:

- Each scheme in the new family is quasi-interpolatory. This guarantees good approximation orders of the schemes.
- Each scheme has a tension parameter so that it provides design flexibility. In particular, with proper tension parameters, it can provide smooth curves without undesirable artifacts even with very irregular control points (see Fig. 3 in Section 5).
- For a suitable range of tension parameter away from zero, the scheme provides the same or better smoothness at the expense of slightly larger support than the Deslauriers–Dubuc schemes and the quadratic and cubic B-splines, when all the schemes are based on the same polynomial space. For instance, the new scheme reproducing cubic polynomials can provide up to C^3-smoothness, while the 4-point Deslauriers–Dubuc scheme is C^1 and the cubic B-spline is C^2 (see Table 3 in Section 5).
- The new schemes provide a large class of refinable functions which generate multiresolution analysis. Thus, the refinable functions obtained by the new schemes can be used for the construction of wavelet systems that balance and meet various demands, such as regularity of wavelets, shapes of refinable functions and approximation power, in time-frequency analysis (H. Kim et al., 2005).

The rest of the paper is organized as follows: Section 2 is devoted to construct a new family of quasi-interpolatory subdivision schemes. We discuss the convergence and the smoothness of the new schemes in Section 3, and the approximation order in Section 4. In Section 5 we illustrate the performance of the new schemes with some specific examples.

2. Construction of the subdivision rule

In this section, we construct a new class of quasi-interpolatory subdivision schemes, which reproduce polynomials up to a certain degree.

Starting with the initial values $f^0 = \{f_n^0 \in \mathbb{R}\colon n \in \mathbb{Z}\}$, our subdivision defines recursively new discrete values $f^k = \{f_n^k \in \mathbb{R}\colon n \in \mathbb{Z}\}$ on finer levels by linear sums of existing values as follows:

$$f_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n} f_n^k, \quad k \in \mathbb{Z}_+. \tag{3}$$

We assume here that only a finite number of coefficients a_n are non-zero so that changes of one control point affect only a limited number of the control points in the next level. This property clearly facilitates the practical implementation of (3). The mask $\{a_n\colon n \in \mathbb{Z}\}$ consisting of the nonzero coefficients is divided into the even and the odd masks corresponding to even and odd n respectively. Their construction depends on the space of polynomials $\Pi_{<L}$ which we want to reproduce. Here $\Pi_{<L}$ stands for the set of all univariate polynomials of degree less than L. Since one desirable property is the symmetry of the basic limit function in (2), we consider the following two cases separately:

Case 1: L is even, i.e., $L =: 2N$.
For the construction of the odd mask, we use the stencil of $L = 2N$ points to reproduce polynomials in $\Pi_{<L}$. That is, the odd mask $\{a_{1-2n}\colon n = -N+1, \ldots, N\}$ is obtained by solving the linear system:

$$p_\ell\left(\frac{1}{2}\right) = \sum_{n=-N+1}^{N} a_{1-2n} p_\ell(n), \quad \ell = 1, \ldots, L, \tag{4}$$

where $p_1, \ldots, p_L$ is a basis of $\Pi_{<L}$. Obviously, there is a unique solution of the linear system (4), and it is exactly the same as the odd mask of the L-point Deslauriers–Dubuc scheme. Next, for the construction of the even mask, we use the stencil of $L + 1 = 2N + 1$ points to reproduce polynomials in $\Pi_{<L}$. That is, the even mask $\{a_{2n}\colon n = -N, \ldots, N\}$ is obtained by solving the linear system:

$$p_\ell(0) = \sum_{n=-N}^{N} a_{-2n} p_\ell(n), \quad \ell = 1, \ldots, L. \tag{5}$$

This is an underdetermined system of $L + 1$ unknowns in L equations so that there is one degree of freedom which will be used as a tension parameter ω. One may set

$$\omega := a_{2N}. \tag{6}$$

If $\omega = 0$, then the scheme becomes the L-point Deslauriers–Dubuc interpolatory scheme, as we will see later in Example 1. Note that the support of the mask is $\mathbb{Z} \cap [-L, \ldots, L]$. As an example, the stencil for the case $L = 4$ is described in Fig. 1(B).

Case 2: L is odd, i.e., $L =: 2N + 1$.
On the purpose of obtaining symmetric rules, we employ $L + 1 = 2(N + 1)$-point scheme reproducing polynomials in $\Pi_{<L}$, and define new (refined) values at $\frac{1}{4}$ and $\frac{3}{4}$ locations between successive old points. Based on this idea, the masks $\{a_{j-2n}\colon n = -N, \ldots, N + 1\}$ with $j = 0, 1$ can be given by solving the linear systems: For the even mask $\{a_{2n}\colon n \in \mathbb{Z}\}$,

$$p_\ell\left(\frac{1}{4}\right) = \sum_{n=-N}^{N+1} a_{-2n} p_\ell(n), \quad \ell = 1, \ldots, L, \tag{7}$$

where $p_1, \ldots, p_L$ is a basis of $\Pi_{<L}$. Next, for the odd mask $\{a_{1-2n}\colon n = -N, \ldots, N + 1\}$,

$$p_\ell\left(\frac{3}{4}\right) = \sum_{n=-N}^{N+1} a_{1-2n} p_\ell(n), \quad \ell = 1, \ldots, L. \tag{8}$$

Then the obtained subdivision rule satisfies the following relation

$$a_{-2n} = a_{2(n-1)+1}, \quad n = -N, \ldots, N + 1,$$

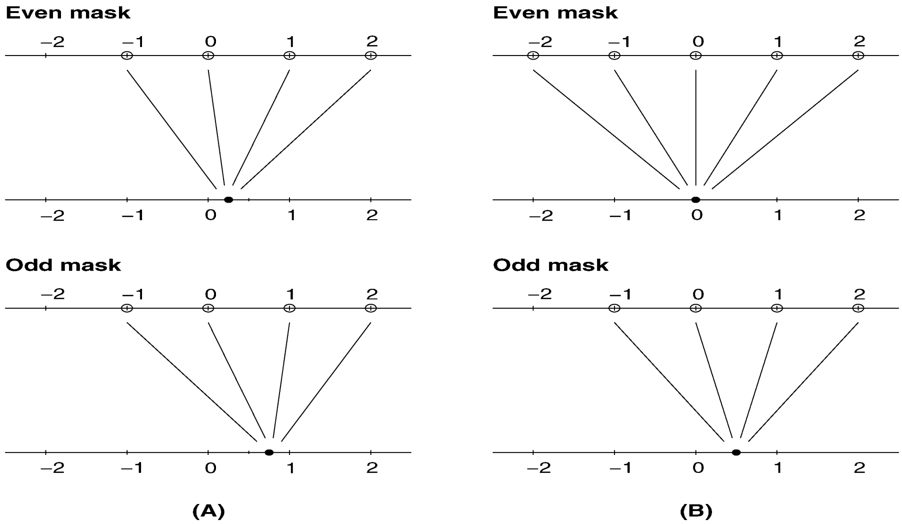


Fig. 1. The construction of masks. (A) and (B) indicate the stencils of the scheme S_L with $L = 3$ and 4 respectively.

and hence, the basic limit function becomes also symmetric. Each of the systems (7) and (8) consists of $L + 1$ unknowns in L equations and eventually has one degree of freedom which is used as the tension parameter ω. One may set $\omega := a_{2N}$ and $\omega := a_{-2N-1}$ for (7) and (8) respectively. The support of the mask is $\mathbb{Z} \cap [-L-1, \ldots, L]$. As an example, the stencil for the case $L = 3$ is described in Fig. 1(A).

Here and in the sequel, for any $L \in \mathbb{N}$, we denote by S_L the quasi-interpolatory subdivision scheme defined as above. The general forms of the masks are presented at Table 1 for $L = 1, \ldots, 10$ with maximal smoothness and the corresponding range of ω. The detailed smoothness depending on ω is specified in Table 4 up to $L = 20$. Now we observe that the $2N$-point Deslauriers–Dubuc scheme and the (quadratic and cubic) B-splines are special cases of our new schemes. The following examples discuss it in detail:

Example 1 *(Deslauriers–Dubuc schemes).* Let $L = 2N$, and for simplicity, set $p_\ell(x) = x^{\ell-1}$ for $\ell = 1, \ldots, L$. From (6), the system (5) becomes of the form:

$$\sum_{n=-N+1}^{N} a_{-2n} n^{\ell-1} = \delta_{\ell-1,0}, \quad \ell = 1, \ldots, L, \tag{9}$$

when $\omega = 0$. This system has $2N$ unknowns in $2N$ equations, and it is easy to see that it has only the trivial solution $a_{2n} = \delta_{n,0}$, $n \in \mathbb{Z}$. Together with the odd mask from (4), this is exactly the mask of the $2N$-point Deslauriers–Dubuc scheme.

Example 2 *(Quadratic and cubic B-spline schemes).* Let $L = 1$. The mask of S_L takes the form $\{\omega, 1-\omega, 1-\omega, \omega\}$. Putting $\omega = \frac{1}{4}$, we have $\frac{1}{4}\{1, 3, 3, 1\}$, which is exactly the mask of the quadratic B-spline scheme. Similarly, we can easily get the mask $\frac{1}{8}\{1, 4, 6, 4, 1\}$ of the cubic B-spline scheme, if we take $L = 2$ and $\omega = \frac{1}{8}$ (see Table 1).

Furthermore, the new four-point subdivision scheme developed by Dyn et al. (2004) can also be obtained by choosing $\omega = 5/128$ in the scheme S_L with $L = 3$. In addition, by this choice of $\omega = 5/128$, the scheme is precise up to cubic polynomials. In fact, for each scheme S_L with L odd, there is a special choice of ω which increases the approximation power. The resulting schemes coincide with the family of schemes introduced by Dyn et al. (2004).

Table 1
The general forms of the masks of S_L. The last two columns indicate the maximum smoothness of S_L and the corresponding ranges of ω, which are obtained by computing $\|(\frac{1}{2}S_L)^{13}\|_\infty < 1$ with MAPLE 8, digits = 15

L	Mask	C^γ	Range of w
1	$[w, 1-w, 1-w, w]$	C^1	$w=\frac{1}{4}$
2	$[w, \frac{1}{2}, 1-2w, \frac{1}{2}, w]$	C^2	$w=\frac{1}{8}$
3	$[-w, -\frac{3}{32}+w, \frac{5}{32}+3w, \frac{15}{16}-3w, \frac{15}{16}-3w, \frac{5}{32}+3w, -\frac{3}{32}+w, -w]$	C^2	$0.030799 < w < 0.085930$
4	$[-w, -\frac{1}{16}, 4w, \frac{9}{16}, 1-6w, \frac{9}{16}, 4w, -\frac{1}{16}, -w]$	C^3	$0.020262 < w < 0.044039$
5	$[w, \frac{35}{2048}-w, -\frac{45}{2048}-5w, -\frac{63}{512}+5w, \frac{105}{512}+10w, \frac{945}{1024}-10w,$ $\frac{945}{1024}-10w, \frac{105}{512}+10w, -\frac{63}{512}+5w, -\frac{45}{2048}-5w, \frac{35}{2048}-w, w]$	C^3	$0.006261 < w < 0.016737$
6	$[w, \frac{3}{256}, -6w, -\frac{25}{256}, 15w, \frac{150}{256}, 1-20w, \frac{150}{256}, 15w, -\frac{25}{256}, -6w, \frac{3}{256}, w]$	C^4	$0.004495 < w < 0.008855$
7	$[-w, -\frac{231}{65536}+w, \frac{273}{65536}+7w, \frac{1001}{32768}-7w, -\frac{1287}{32768}-21w, -\frac{9009}{65536}+21w,$ $\frac{15015}{65536}+35w, \frac{15015}{16384}-35w, \frac{15015}{16384}-35w, \frac{15015}{65536}+35w,$ $-\frac{9009}{65536}+21w, -\frac{1287}{32768}-21w, \frac{1001}{32768}-7w, \frac{273}{65536}+7w, -\frac{231}{65536}+w, -w]$	C^4	$0.001598 < w < 0.003228$
8	$[-w, -\frac{5}{2048}, 8w, \frac{49}{2048}, -28w, -\frac{245}{2048}, 56w, \frac{1225}{2048}, 1-70w,$ $\frac{1225}{2048}, 56w, -\frac{245}{2048}, -28w, \frac{49}{2048}, 8w, -\frac{5}{2048}, -w]$	C^5	$0.001132 < w < 0.001754$
9	$[w, \frac{6435}{8388608}-w, -\frac{7293}{8388608}-9w, -\frac{8415}{1048576}+9w, \frac{9945}{1048576}+36w,$ $\frac{85085}{2097152}-36w, -\frac{109395}{2097152}-84w, -\frac{153153}{1048576}+84w,$ $\frac{255255}{1048576}+126w, \frac{3828825}{4194304}-126w, \frac{3828825}{4194304}-126w, \frac{255255}{1048576}+126w,$ $-\frac{153153}{1048576}+84w, -\frac{109395}{2097152}-84w, \frac{85085}{2097152}-36w,$ $\frac{9945}{1048576}+36w, -\frac{8415}{1048576}+9w, -\frac{7293}{8388608}-9w, \frac{6435}{8388608}-w, w]$	C^5	$0.000394 < w < 0.000629$
10	$[w, \frac{35}{65536}, -10w, -\frac{405}{65536}, 45w, \frac{567}{16384}, -120w, -\frac{2205}{16384}, 210w,$ $\frac{19845}{32768}, -252w+1, \frac{19845}{32768},$ $210w, -\frac{2205}{16384}, -120w, \frac{567}{16384}, 45w, -\frac{405}{65536}, -10w, \frac{35}{65536}, w]$	C^6	$0.000301 < w < 0.000355$

3. Convergence and smoothness analysis

With the mask $\{a_n\}$ of S_L at hand, a fundamental question will be the convergence and the smoothness of S_L, which will be the main topic of this section. To simplify the presentation of subdivision schemes and their analysis, it is convenient to assign to each rule $\{a_n\colon n \in \mathbb{Z}\}$ the *Laurent polynomial*

$$a(z) := \sum_{n\in\mathbb{Z}} a_n z^n,$$

where only a finite number of the coefficients a_n are non-zero. Further, the Laurent polynomial corresponding to the m-iterated rule S_L^m is given by

$$a^{[m]}(z) = \prod_{\ell=0}^{m-1} a\left(z^{2^\ell}\right) = \sum_{n\in\mathbb{Z}} a_n^{[m]} z^n, \tag{10}$$

where the scheme corresponding to $\{a_n^{[m]}\}$ is a rule mapping f^k to f^{k+m}. Note that the norm of S_L^m is

$$\|S_L^m\|_\infty = \max\left\{\sum_{n\in\mathbb{Z}} \left|a_{j+2^m n}^{[m]}\right|\colon\ j = 0, 1, \ldots, 2^m - 1\right\}.$$

Let $S_{L,1}$ be the subdivision rule for the divided differences of the original control points so that it has the property

$$df^{k+1} = S_{L,1} df^k,$$

Table 2
Let $L = 2N$. Comparison between the (maximal) smoothness of S_L and Deslauriers–Dubuc (DD) interpolatory schemes

L	2	4	6	8	10	12	14	16	18	20
DD	0	1	2	3	4	4	5	5	6	6
S_L	2	3	4	5	6	6	7	7	8	9

where $f^k = S_L^k f^0$ and $(df^{k+1})_n = 2^k(f_n^k - f_{n-1}^k)$. In fact, the characteristic Laurent polynomial $a_1(z)$ for $S_{L,1}$ is given by

$$a_1(z) = \frac{2z}{1+z}a(z).$$

We will use the following standard tools for the convergence analysis of subdivision schemes:

Theorem 1. (Dyn, 1992) *Let S and S_1 be stationary subdivision schemes with the characteristic Laurent polynomials $a(z)$ and $a_1(z)$, respectively. Then, we have the following*:

(1) *The scheme S is uniformly convergent if and only if there exists an integer $M \geqslant 1$ such that $\|(\frac{1}{2}S_1)^M\|_\infty < 1$.*
(2) *If $a(z) = \frac{1}{2}(1+z)b(z)$ and the scheme S_b corresponding to $b(z)$ is C^γ, then S is a uniformly convergent scheme and the basic limit function φ of S is in $C^{\gamma+1}(\mathbb{R})$.*

We can induce from Theorem 1 that in order to assure a certain smoothness of S_L, the Laurent polynomial should have the (smoothing) factor $2^{-\ell}(1+z)^\ell$ for some $\ell \in \mathbb{N}$. Indeed the polynomial reproducing property of S_L as in (4), (5), (7) and (8) guarantees the existence of this smoothing factor such that

$$a(z) = 2^{-L}(l+z)^L q(z) \tag{11}$$

for some Laurent polynomial $q(z)$. For more details about the above discussion, the readers are referred to the paper (Dyn, 1992).

Recall that if $\omega = 0$ and L is even, the scheme S_L becomes the L-point Deslauriers–Dubuc scheme. Thus, for a small perturbation of ω around zero, we can prove immediately that the scheme S_L has at least the same smoothness as the L-point Deslauriers–Dubuc scheme.

Theorem 2. *Let L be an even integer, i.e., $L = 2N$ with $N \in \mathbb{N}$, and assume that the L-point Deslauriers–Dubuc scheme is C^γ with $\gamma \in \mathbb{N}$. Then, there exists $\omega_0 > 0$ such that for any $|\omega| < \omega_0$, the scheme S_L is also C^γ.*

Beyond the observation in the above theorem, the smoothness of S_L can be increased by choosing ω from a suitable area away from the origin. The specific smoothness of S_L can be obtained by using Theorem 1. Although this is algorithmic in principle it is almost impossible to analyze it without the help of a computer program because the algebraic manipulations are too much involved. Thus, the MAPLE program is utilized to figure out the convergence and smoothness of S_L. The readers who are interested in knowing the details about this algorithm are referred to the paper (Dyn, 1992). Further, it is also necessary to remark that the support of the basic limit function of S_L is $[-L, L]$, which is slightly larger than the case of the L-point Deslauriers–Dubuc scheme, i.e., $[-L+1, L-1]$. Table 2 provides the comparison between the smoothness of S_L and the Deslauriers–Dubuc interpolatory scheme.

As we have seen in Example 2, S_L becomes the quadratic B-spline scheme when $L = 1$ and $\omega = \frac{1}{4}$, and the cubic B-spline scheme when $L = 2$ and $\omega = \frac{1}{8}$.

4. Approximation order

An important issue in the implementation of subdivision algorithm is how to actually attain the original function as close as possible if a given initial data f^0 is sampled from an underlying function. A high quality reconstruction scheme should guarantee that the approximation error decreases when the quality of the sample increases.

For simplicity, suppose that $f^0 = \{f_n^0\colon n \in \mathbb{Z}\}$ is sampled from an underlying function f with the density 2^{-k_0} for some $k_0 \in \mathbb{Z}$. Then our goal is to find the largest exponent $m > 0$ such that

$$\|S^\infty f^0 - f\|_{L_\infty(K)} \leqslant C2^{-k_0 m}$$

with a constant $C > 0$ independent of k_0, where K is a compact subset of $\mathbb{R}$. The exponent m is called the *approximation order* of the scheme S. Furthermore, if a scheme S is uniformly convergent, its limit function can be written as

$$S^\infty f^0 = \sum_{n\in\mathbb{Z}} f_n^0 \varphi(2^{k_0} \cdot - n),$$

where φ is the basic limit function of S defined by $\varphi = S^\infty \delta$ with $\delta = \{\delta_{0,n}\}$ (see (2)). In this paper, we are particularly interested in approximating functions f in the Sobolev space

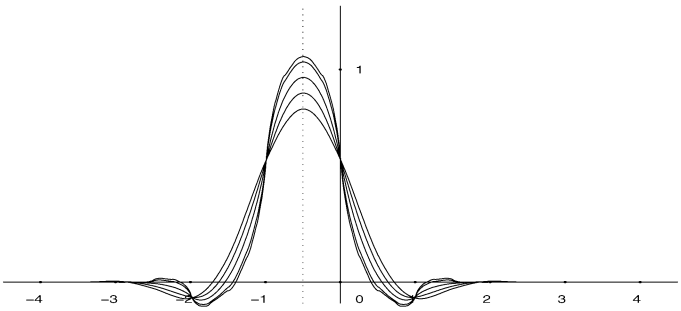
$$W_\infty^n(K) := \left\{ g\colon \sum_{m=0}^{n} \|g^{(m)}\|_{L_\infty(K)} < \infty \right\}, \quad n \in \mathbb{Z}_+.$$

Theorem 3. *Suppose that $f \in W_\infty^L(K)$ and the given initial data $f^0 = \{f_n^0\colon n \in \mathbb{Z}\}$ is of the form: $f_n^0 = f(2^{-k_0} n)$ if L is even, and $f_n^0 = f(2^{-k_0}(n - 1/2))$ if L is odd. Then, for any compact set K in $\mathbb{R}$, we have*

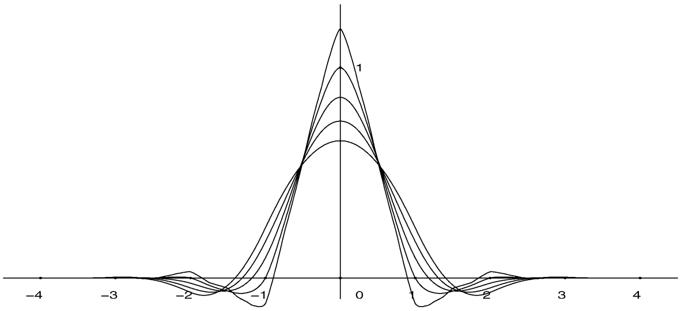
$$\|S^\infty f^0 - f\|_{L_\infty(K)} \leqslant C2^{-k_0 L} \|f^{(L)}\|_{L_\infty(K)}$$

with a constant $C > 0$ independent of k_0.

Proof. First, for the case L is even, the approximation order L can be obtained by using the known technique of quasi-interpolation; for example, see (Powell, 1992, Section 3.8). Next, consider the case L is odd. If the data is sampled from a polynomial in $\Pi_{<L}$, the limit function is a shifted version of the same polynomial, that is, $S_L^\infty p^0 = p$



(a) $L = 3$



(b) $L = 4$

Fig. 2. The effect of the tension ω on the shape of the basic limit functions of S_L with $L = 3, 4$. Here, for (a), $\omega = -0.007, 0, 0.02, 0.04, 0.06$ from the top at $x = -1/2$ and for (b), $\omega = -0.02, 0, 0.02, 0.04, 0.06$ from the top at the origin.

if $p_i^0 = p(i - \frac{1}{2})$ with $p \in \Pi_{<L}$ (Dyn et al., 2004, Lemma 1). Using the same analysis of Theorem 2 in (Dyn et al., 2004)) based on this "shifted" polynomial precision, the scheme S_L can be proven to provide the approximation order L. □

5. Examples

In this section, we illustrate the performance of S_L with some numerical examples. In order to investigate how the tension parameter ω affects the limit function, we first look at the basic limit function φ of S_L (see (2)). Fig. 2 presents

Table 3
Comparison of basic limit functions of cubic B-spline, the 4-point Deslauriers–Dubuc (DD) scheme, and the scheme S_L with $L = 4$

	Cubic B-spline	4-point DD scheme	S_L $(L = 4)$
Support of φ	$[-2, 2]$	$[-3, 3]$	$[-4, 4]$
Maximal smoothness	C^2	C^1	C^3
Approximation order	2	4	4

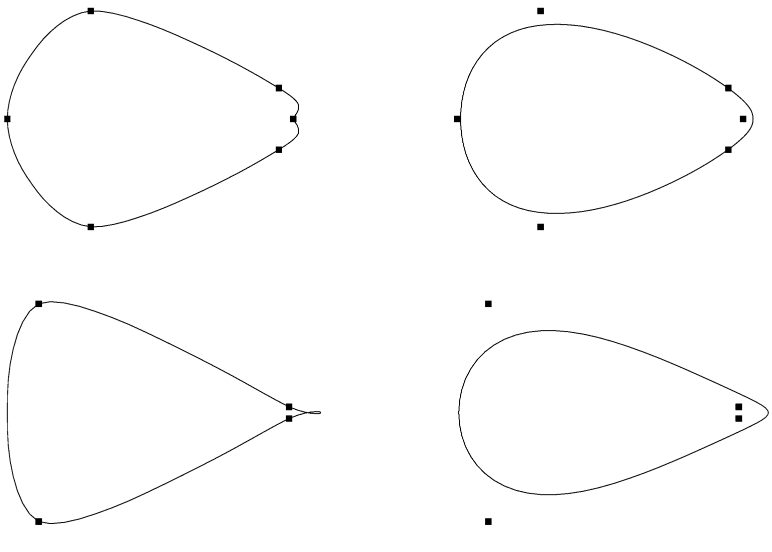


Fig. 3. (A) The 4-point Deslauriers–Dubuc scheme. (B) The scheme S_L with $L = 4$ and $\omega = 0.03$.

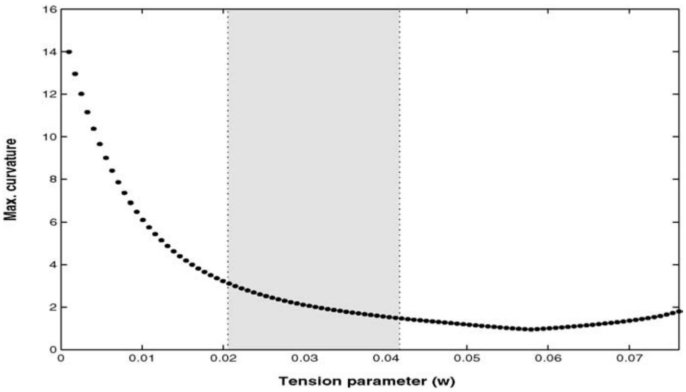


Fig. 4. The curvatures of the basic limit functions of S_L with $L = 4$ and $0 < \omega < 0.0769$, where S_L is C^2. The spot between two vertical dotted lines indicates the area of C^3 smoothness.

Table 4
Smoothness of the scheme S_L. Depending on the range of the tension parameter ω, we can get different smoothness. By computing $\|(\frac{1}{2}S_L)^{13}\|_\infty < 1$ with MAPLE 8, digits = 15, the ranges of ω are obtained

L	1	2	3	4
C^0	.0, .95574250	−.25, .75	−.09247323, .26960141	−.13528135, .24451083
C^1	$\frac{1}{4}$	.0, .47779109	−.01114900, .14653826	−.04005134, .13879261
C^2		$\frac{1}{8}$	.03079863, .08593042	.0, .07693204
C^3				.02026201, .04403884

L	5	6	7	8
C^0	−.04612298, .08273090	−.05539807, .07863532	−.01800369, .02613960	−.02010382, .02534770
C^1	−.01680405, .04509990	−.02119504, .04252798	−.00803940, .01409678	−.00851685, .01336099
C^2	.00004704, .02639276	−.00668461, .02363848	−.00208118, .00809265	−.00358493, .00732443
C^3	.00626104, .01673668	.00160036, .01438940	.00036845, .00501444	−.00061206, .00436987
C^4		.00449500, .00885490	.00159808, .00322807	.00056945, .00282662
C^5				.00113172, .00175372

L	9	10	11	12
C^0	−.00644494, .00837536	−.00693486, .00823911	−.00222275, .00270558	−.00234159, .00269391
C^1	−.00311559, .00443643	−.00309893, .00425457	−.00110817, .00141017	−.00107919, .00136704
C^2	−.00110257, .00248603	−.00144914, .00228808	−.00044613, .00076978	−.00052675, .00072123
C^3	−.00025920, .00149172	−.00044421, .00132523	−.00016482, .00044674	−.00019655, .00040501
C^4	.00022995, .00102248	−.00002958, .00082733	.00000460, .00029266	−.00005715, .00024349
C^5	.00039388, .00062908	.00020669, .00059615	.00007065, .00020936	.00002598, .00016755
C^6		.00030131, .00035479	.00011318, .00012718	.00005728, .00012607

L	13	14	15	16
C^0	−.00075324, .00088073	−.00078420, .00088323	−.00025275, .00028819	−.00026176, .00029022
C^1	−.00038211, .00045317	−.00036775, .00044298	−.00012972, .00014667	−.00012401, .00014439
C^2	−.00016417, .00024162	−.00018382, .00022985	−.00005783, .00007688	−.00006299, .00007392
C^3	−.00007013, .00013559	−.00007545, .00012550	−.00002656, .00004187	−.00002721, .00003950
C^4	−.00001440, .00008479	−.00002892, .00007260	−.00000827, .00002500	−.00001174, .00002205
C^5	.00000867, .00005769	−.00000129, .00004763	−.00000050, .00001617	−.00000268, .00001377
C^6	.00002154, .00004497	.00000997, .00003405	.00000401, .00001205	.00000115, .00000935
C^7		.00001636, .00002352	.00000591, .00000870	.00000333, .00000730
C^8				.00000440, .00000480

L	17	18	19	20
C^0	−.00008447, .00009455	−.00008709, .00009559	−.00002813, .00003112	−.00002892, .00003151
C^1	−.00004371, .00004774	−.00004155, .00004726	−.00001462, .00001561	−.00001387, .00001552
C^2	−.00001994, .00002472	−.00002134, .00002394	−.00000679, .00000800	−.00000717, .00000780
C^3	−.00000955, .00001313	−.00000955, .00001259	−.00000334, .00000417	−.00000327, .00000405
C^4	−.00000352, .00000754	−.00000438, .00000682	−.00000135, .00000233	−.00000156, .00000214
C^5	−.00000093, .00000464	−.00000140, .00000407	−.00000047, .00000137	−.00000058, .00000123
C^6	.00000058, .00000328	−.00000012, .00000262	.00000001, .00000091	−.00000016, .00000076
C^7	.00000120, .00000253	.00000062, .00000195	.00000023, .00000067	.00000009, .00000053
C^8	.00000158, .00000173	.00000093, .00000157	.00000034, .00000055	.00000019, .00000041
C^9				.00000025, .00000031

various basic limit functions depending on the tension $\omega = -0.007, 0, 0.02, 0.04, 0.06$ with $L = 3$ and $\omega = -0.02, 0, 0.02, 0.04, 0.06$ with $L = 4$. When $\omega = 0$, it becomes interpolatory. In Table 3, we compare the scheme S_L, $L = 4$, with the cubic B-spline and the 4-point Deslauriers–Dubuc scheme.

Fig. 3 shows an advantage of using S_L. In fact, if the given control points are very irregular, the limit curves of the Deslauriers–Dubuc schemes may result in unpleasant artifacts as shown in Fig. 3(A). However, choosing an ω away from the origin, we can obtain visually better curves without twisting artifacts. Having performed numerical experimentations with several alternatives for ω, we found out that a good choice is about $0.025 \leqslant \omega \leqslant 0.06$. Fig. 4 describes the maximum curvatures of φ with $L = 4$, corresponding to ω in the area of C^2 smoothness.

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