Approximation by Conditionally Positive Definite Functions with Finitely Many Centers

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Abstract. The theory of interpolation by using conditionally positive definite function provides optimal error bounds when the basis function ϕ is smooth and the approximant f is in a certain native space $\mathcal{F}_\phi$. The space $\mathcal{F}_\phi$, however, is very small for the case where ϕ is smooth. Hence, in this study, we are interested in the approximation power of interpolation to mollifications of functions in Sobolev space. Specifically, it turns out that interpolation to mollifications provides spectral error bounds depending only on the smoothness of the functions f.

§1. Introduction

In the last decades or so, there has been considerable progress concerning the scattered data approximation problem in two or more dimensions. In particular, the methods of radial basis function approximation are becoming increasingly popular. Usually, the starting point of the approximation process is the choice of a conditionally positive definite function $\phi : \mathbb{R}^d \to \mathbb{R}$.

Definition 1.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We say that ϕ is conditionally positive definite of order $m \in \mathbb{N} := \{1, 2, \cdots\}$ if for every finite set of pairwise distinct points $X := \{x_1, \cdots, x_N\} \subset \mathbb{R}^d$ and for every $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N \setminus 0$ satisfying*

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0, \quad p \in \Pi_m,$$

the quadratic form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(x_i - x_j)$$

Trends in Approximation Theory
Kirill Kopotun, Tom Lyche, and Mike Neamtu (eds.), pp. 1–10.

ISBN 0-8265-xxxx-x.

is positive. Here, Π_m denotes the subspace of $C(\mathbb{R}^d)$ consisting of all algebraic polynomials of degree less than m on $\mathbb{R}^d$.

Given a conditionally positive definite function $\phi : \mathbb{R}^d \to \mathbb{R}$, we consider approximation of a function $f : \mathbb{R}^d \to \mathbb{R}$ by linear combinations of finitely many translates $\phi(\cdot - x_j)$, $j = 1, \cdots, N$, of ϕ. In many cases, it is helpful to have additional polynomial terms. Then, we write the approximants as

$$a_{f,X}(x) := \sum_{i=1}^{\ell} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi(x - x_j), \tag{1.1}$$

where $p_1, \cdots, p_\ell$ is a basis for Π_m and α_j, $j = 1, \cdots, N$, are chosen so that

$$\sum_{j=1}^{N} \alpha_j q(x_j) = 0 \quad \text{for all} \quad q \in \Pi_m.$$

In particular, $a_{f,X}$ becomes the ϕ-interpolant when $a_{f,X}$ satisfies the conditions

$$a_{f,X}(x_j) = f(x_j), \quad j = 1, \cdots, N.$$

The strengths of this method are as follows: (i) the accuracy of approximation is usually very satisfactory provided the approximand f is reasonably smooth; (ii) there is enough flexibility in the choice of basis functions. The common choices of ϕ include:

(a) $\phi(x) := (-1)^{\lceil m-d/2 \rceil}(|x|^2 + \lambda^2)^{m-d/2}$, d odd, $m > d/2$, (multiquadrics),

(b) $\phi(x) := (-1)^{m-d/2+1}(|x|^2 + \lambda^2)^{m-d/2} \log(|x|^2 + \lambda^2)^{1/2}$, $m > d/2$, d even, ('shifted' surface splines).

(c) $\phi(x) := (|x|^2 + \lambda^2)^{m-d/2}$, $0 < m < d/2$, (inverse multiquadrics),

(d) $\phi(x) := \exp(-\alpha|x|^2)$, $\alpha > 0$, (Gaussians).

where d, $m \in \mathbb{N}$ and $\lambda > 0$, and where $\lceil s \rceil$ indicates the smallest integer greater than s. Indeed, the positive definiteness of continuous and absolutely integrable functions $\phi : \mathbb{R}^d \to \mathbb{R}$ is equivalent to the Fourier transform $\hat{\phi}$ being nonnegative on $\mathbb{R}^d$ and positive at least on an open subset of $\mathbb{R}^d$. We note in passing that this argument is a consequence of the simple identity

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \varphi(x_j - x_k) = \int_{\mathbb{R}^d} \hat{\varphi}(\theta) \Big| \sum_{j=1}^{N} \alpha_j e^{i x_j \cdot \theta} \Big|^2 d\theta$$

for any $(\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N \setminus 0$ and the fact that the map $\theta \mapsto \sum_{j=1}^{N} \alpha_j e^{i x_j \cdot \theta}$, $\theta \in \mathbb{R}^d$, has zeros at most on a set of measure zero. This identity is also generalized to the case of conditionally positive definite functions ϕ of order $m > 0$. Such conditionally positive definite functions obviously exist in abundance. For instance, by the convolution theorem, all $2n \geq 2$-fold convolutions of compactly supported functions on $\mathbb{R}^d$ with Fourier transforms that are nonzero at least on an open subset of $\mathbb{R}^d$ will have strictly nonnegative Fourier transform and thus be conditionally positive definite. Among all those functions, the present paper is mainly concerned with using radially symmetric functions but not necessary restricted to this condition. We assume a function ϕ to be radial in the sense that $\phi(x) = \Phi(|x|)$ and to be of at most polynomial growth at infinity.

For a given basis function ϕ whose (tempered) Fourier transform coincide on $\mathbb{R}^d \setminus 0$ with a positive continuous function, the existing theory of interpolation estimates errors for the functions in the space

$$\mathcal{F}_\phi := \{g : |g|_\phi := \int_{\mathbb{R}^d} \frac{|\hat{g}(\theta)|^2}{\hat{\phi}(\theta)} d\theta < \infty\}$$

which is called "native" function space ([8], [14]). Specifically, for all $x \in \Omega$ and $f \in \mathcal{F}_\phi$, bounds for the interpolation error are of the form

$$|f(x) - a_{f,X}(x)| \le P_{\phi,X}(x)|f|_\phi.$$

Here $P_{\phi,X}$ is the *power function* that evaluates the norm of the error functional:

$$P_{\phi,X}(x) = \sup_{|f|_\phi \neq 0} \frac{|f(x) - a_{f,X}(x)|}{|f|_\phi}.$$

In particular, when the basis function ϕ is smooth (e.g, Gaussian), the interpolation method provides optimal asymptotic decay of errors. The space $\mathcal{F}_\phi$, however is very small ([8], [14]). The approximands f need to be extremely smooth for effective error estimates. Thus, employing smooth basis functions ϕ, our major concern is to study the approximation power of interpolation to mollified functions of f which belongs to larger spaces, especially to the Sobolev spaces. For any $k \in \mathbb{N}$, we define the Sobolev spaces by

$$W_p^k(\mathbb{R}^d) := \{f : |f|_{k,p} := \sum_{|\alpha|_1 \le k} \|D^\alpha f\|_{L_p(\mathbb{R}^d)} < \infty\}$$

with $1 \le p \le \infty$.

In [13], Schaback also studied the interpolation behavior to mollifications of a function f. In this paper, however, we provide sharper error estimates than [13]. Another advantage of this study is that it can be applied to a wider range of basis functions ϕ. We can employ any basis function ϕ whose Fourier transform $\hat{\phi}$ is nonnegative on $\mathbb{R}^d$ and positive on an open ball centered at the origin in $\mathbb{R}^d$.

§2. Basic Assumptions

Let $\mathcal{K}$ be the space of C_0^∞ functions with the topology in [5], and let $\mathcal{S}$ be the space of rapidly decaying functions. Throughout this paper, the function ϕ is considered as a tempered distribution on $\mathcal{K}$, and we assume that its Fourier transform $\hat{\phi}$ coincides on $\mathbb{R}^d \setminus 0$ with some continuous function while having a certain type of singularity (necessarily of a finite order) at the origin, i.e., $\hat{\phi}$ is of the form

$$|\cdot|^{2m}\hat{\phi} = F \tag{2.1}$$

with $m \geq 0$ and F a nonnegative function in $\mathcal{S}$. In particular, we assume that the function F is positive on the Euclidean ball B_r centered at the origin for some $r > 0$.

Assuming $\phi = \Phi(|\cdot|)$ to be conditionally positive definite of order $m \geq 0$, we require X to have the nondegeneracy property for Π_m, that is, if $p \in \Pi_m$ and $p|_X = 0$, then $p = 0$. We also assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain with cone property. Then, for a given set X in Ω, we measure the 'density' of X by

$$h := h(X;\Omega) := \sup_{x\in\Omega} \min_{x_j\in X} |x - x_j|.$$

For a given basis function ϕ and a center set X, we especially adopt the scaled basis function

$$\phi_\omega := \phi(\cdot/\omega)$$

where

$$\omega := \omega(h)$$

is a parameter depending on h such that $h/\omega \to 0$ as h tends to 0. Then, in order to differentiate from $a_{f,X}$ in (1.1), we use the notation

$$s_{f,X}(x) := \sum_{i=1}^{\ell} \beta_i p_i(x) + \sum_{j=1}^{N} \alpha_j \phi_\omega(x - x_j), \tag{2.2}$$

where $p_1, \cdots, p_\ell$ is a basis for Π_m.

§3. Error Estimates

For a given continuous function f, we first approximate f by a mollified function $f^*_\omega \in \mathcal{F}_\phi$ obtained via truncation of the Fourier transform, i.e.,

$$f^*_\omega := \sigma(\cdot/\omega)^\vee * f$$

where

$$\sigma : \mathbb{R}^d \to [0,1]$$

is a nonnegative C^∞-cutoff function whose support σ lies in the Euclidean ball B_η with $\sigma = 1$ on $B_{\eta/2}$ and $\|\sigma\|_{L_\infty(\mathbb{R}^d)} = 1$. Here and in the sequel, we assume that $\eta < r$, where r appears in the property of F in (2.1). Then, in this section, we will observe the approximation behavior of $s_{f^*_\omega,X}$ to f.

The following lemma is from [15].

Lemma 3.1. *Let $\omega = \omega(h)$ be a parameter depending on h the density of X. Then, for every $f \in W^k_\infty(\mathbb{R}^d)$ with $k > 0$, we have the following decay property*

$$\|f - \sigma^\vee_\omega * f\|_\infty = o(\omega^k).$$

From the papers ([8], [14]), we cite

Lemma 3.2. *Let $a_{X,f}$ in (1.1) be an interpolant to f on $X = \{x_1, \cdots, x_N\}$. Given ϕ and m, for all functions f in the native space $\mathcal{F}_\phi$, there is an error bound of the form*

$$|f(x) - a_{f,X}(x)| \le |f|_\phi P_{\phi,X}(x)$$

where $P_{\phi,X}(x)$ is the norm of the error functional, i.e.,

$$P_{\phi,X}(x) = \sup_{|f|_\phi \neq 0} \frac{|f(x) - a_{f,X}(x)|}{|f|_\phi} \tag{3.1}$$

and it is the minimum of all such norms, if quasi-interpolants

$$q_{f,X}(x) := \sum_{j=1}^N u_j(x) f(x_j)$$

with $p(x) = \sum_{j=1}^N u_j(x)p(x_j)$, $p \in \Pi_m$, are allowed instead of $a_{f,X}$.

We now present the following theorem.

Theorem 3.3. *Let $s_{f^*_\omega,X}$ in (2.2) be an interpolant to f^*_ω on X by way of employing $\phi_\omega = \phi(\cdot/\omega)$. Suppose that the Fourier transform $\hat\phi$ of ϕ satisfies the condition (2.1). Then, for every function $f \in W^k_\infty(\mathbb{R}^d) \cap W^q_2(\mathbb{R}^d)$ with $q = \min(k, m)$, we have*

$$|f(x) - s_{f^*_\omega,X}(x)| \le o(\omega^k) + c\omega^{q-d/2} P_{\phi,X/\omega}(x/\omega)|f|_{k,2}, \quad x \in \Omega,$$

with a constant $c > 0$ independent of X and Ω. Here, m is the order of singularity of $\hat\phi$ at the origin.

Proof. We first split the error $f - s_{f^*_\omega,X}$ by the two terms:

$$f - s_{f^*_\omega,X} = (f^*_\omega - s_{f^*_\omega,X}) + (f - f^*_\omega).$$

Since $\|f - f^*_\omega\|_{L_\infty(\mathbb{R}^d)} = o(\omega^k)$ by Lemma 3.1, it suffices to estimate only the error $f^*_\omega - s_{f^*_\omega,X}$. From the definition of $s_{f^*_\omega,X}$ in (2.2), we can write

$$s_{f^*_\omega,X}(\omega\cdot) := \sum_{i=1}^{\ell} \beta_i p_i(\omega\cdot) + \sum_{j=1}^N \alpha_j \phi(\cdot - x_j/\omega),$$

and then it is obvious that

$$s_{f^*_\omega,X}(\omega\cdot)|_{X/\omega} = f^*_\omega(\omega\cdot)|_{X/\omega}.$$

Thus, one simply notes that the function $s_{f_\omega^*,X}(\omega\cdot)$ can be considered as an interpolant (employing the translates $\phi(\cdot - x_j),\ j = 1,\cdots,N$) to the dilated function $f_\omega^*(\omega\cdot)$ on X/ω, i.e.,

$$s_{f_\omega^*,X}(\omega\cdot) = a_{f_\omega^*(\omega\cdot),X/\omega}.$$

Since the function $f_\omega^*(\omega\cdot)$ belongs to the native space $\mathcal{F}_\phi$, it is immediate from Lemma 3.2 that, for any $x \in \Omega$,

$$\begin{aligned}|f_\omega^*(x) - s_{f_\omega^*,X}(x)| &= |f_\omega^*(\omega\cdot) - a_{f_\omega^*(\omega\cdot),X/\omega}|(x/\omega)\\ &\le P_{\phi,X/\omega}(x/\omega)|f_\omega^*(\omega\cdot)|_\phi.\end{aligned}$$

Here, let us first consider the case $k \ge m$. Using the condition $|\cdot|^{2m}\hat{\phi} = F$ with $F \in \mathcal{S}$, the explicit formula of the norm $|\cdot|_\phi$ induces

$$\begin{aligned}|f_\omega^*(\omega\cdot)|_\phi^2 &\le \omega^{-2d}\int_{B_\eta} \big|\sigma(\theta)\hat{f}(\theta/\omega)\big|^2 \hat{\phi}^{-1}(\theta)d\theta\\ &= \omega^{2m-d}\int_{B_\eta/\omega} \big(\frac{\sigma^2}{F}\big)(\omega\theta)\big||\cdot|^{2m}\hat{f}\big|^2(\theta)d\theta\\ &\le c\omega^{2m-d}\|\sigma^2/F\|_{L_\infty(B_\eta)}|f|_{m,2}^2.\end{aligned}$$

In a similar fashion, for the case $k < m$, we have the bound

$$\begin{aligned}|f_\omega^*(\omega\cdot)|_\phi^2 &\le c\omega^{2k-d}\int_{B_\eta/\omega} |\omega\theta|^{2(m-k)}\big(\frac{\sigma^2}{F}\big)(\omega\theta)\big||\cdot|^{k}\hat{f}\big|^2(\theta)d\theta\\ &\le c\omega^{2k-d}\|\sigma^2|\cdot|^{2(m-k)}/F\|_{L_\infty(B_\eta)}|f|_{k,2}^2.\end{aligned}$$

It completes the proof. $\square$

§4. Approximation in Sobolev Spaces

In this section, employing smooth conditionally positive definite functions ϕ, we prove that the interpolant $s_{f_\omega^*,X}$ provides spectral approximation orders (i.e., the asymptotic rates of the error $f - s_{f_\omega^*,X}$ depend only on the smoothness of the functions f) under some suitable conditions on ω. For this proof, we first estimate the function $P_{\phi,X/\omega}(\cdot/\omega)$ on Ω. In fact, the general idea of our analysis of $P_{\phi,X/\omega}(\cdot/\omega)$ is similar to the work of Wu and Schaback [14]. Our method is simpler, however, and the conditions are less restrictive.

Lemma 4.1. *Let the basis function ϕ satisfy the assumption in* (2.1). *Then, for any $n \in \mathbb{N}$, there exists a constant c_n independent of X and Ω such that*

$$P_{\phi,X/\omega}(x/\omega) \le c_n(h/\omega)^n, \quad x \in \Omega.$$

Proof. Let us denote $u(x) := (u_1(x),\cdots,u_N(x))$ as a vector in $\mathbb{R}^N$. Then, the so-called power function in (3.1) can be rewritten as

$$P_{\phi,X}^2(x) = \min_{u\in K_m}\int_{\mathbb{R}^d} \hat{\phi}(\theta)\big|e^{ix\cdot\theta} - \sum_{j=1}^N u_j(x)e^{ix_j\cdot\theta}\big|^2 d\theta$$

where K_n, $n \in \mathbb{N}$, indicates the set

$$K_n := \big\{(u_1(x), \cdots, u_N(x)) \in \mathbb{R}^N \mid \sum_{j=1}^{N} u_j(x)p(x_j) = p(x),\, p \in \Pi_n\big\},$$

see [14] for the details. Then, for any $n \in \mathbb{N}$ with $n \geq m$, there is a vector $\bar{u} := \bar{u}_n := (\bar{u}_1, \cdots, \bar{u}_N)$ in the admissible set K_n such that it satisfies the following conditions:

(a) There exists $c_1 > 0$ such that, for any $x \in \Omega$, $\bar{u}_j(x) = 0$ whenever $|x - x_j| > c_1 h$, with h the density of X.

(b) The set $\{(\bar{u}_1(x), \cdots, \bar{u}_N(x)) : x \in \Omega\}$ is bounded in $\ell_1(X)$.

For the examples of such vectors $\bar{u}$, the readers are referred to the papers [6] and [15]. Remembering the condition for $\hat{\phi}$ in (2.1), we have

$$P^2_{\phi,X/\omega}(x/\omega) \leq \int_{\mathbb{R}^d} F(\theta)|\theta|^{-2m}\big|1 - \sum_{j=1}^{N} \bar{u}_j(x)e^{i(x_j - x)\cdot\theta/\omega}\big|^2 d\theta. \tag{4.1}$$

Let $p_{n-1}(x)$ be the Taylor expansion of e^x about the origin of degree $n-1$. The polynomial reproducing property of $\bar{u} \in K_n$ implies that

$$\sum_{j=1}^{N} \bar{u}_j(x)[1 - p_{n-1}(i(x_j - x)\cdot\theta/\omega)] = 0.$$

Thus, using the properties (a) and (b) of the vector $\bar{u}$, it follows that

$$\begin{aligned}|\theta|^{-m}\big|1 - \sum_{j=1}^{N} \bar{u}_j(x)e^{i(x_j-x)\cdot\theta/\omega}\big| &\leq |\theta|^{-m}\sum_{j=1}^{N} |\bar{u}_j(x)((x_j - x)\cdot\theta/\omega)^n|/n! \\ &\leq c_n|\theta|^{n-m}h^n/\omega^n\|\bar{u}\|_1\end{aligned}$$

where $\|\bar{u}\|_1$ indicates the ℓ_1-norm of the vector $\bar{u}$. Inserting this bound into (4.1), we get

$$P^2_{\phi,X/\omega}(x/\omega) \leq c'_n(h/\omega)^{2n}\|(\,)^{2(n-m)}F\|_{L_1(\mathbb{R}^d)}.$$

The last integral of the above expression is finite because $F \in \mathcal{S}$. Therefore, we finish the proof. $\square$

Corollary 4.2. *Let ϕ be a smooth basis function satisfying the condition in* (2.1), *i.e., $\hat{\phi}$ is of the form $|\cdot|^{2m}\hat{\phi} = F \in \mathcal{S}$ with $m \geq 0$. Let ω be chosen to satisfy the relation $\omega(h) = h^r$ with $r \in (0,1)$. Then, for any $f \in W^k_\infty(\mathbb{R}^d) \cap W^q_2(\mathbb{R}^d)$ with $q = \min(k,m)$, we have the error bound*

$$|f(x) - s_{f^*_\omega,X}(x)| = o(h^{rk}), \quad x \in \Omega.$$

Proof. It is obvious from Lemma 4.1 that for any $n \in \mathbb{N}$, there is a constant c_n such that

$$P_{\phi,X/\omega}(x/\omega) \leq c_n h^{n(1-r)}$$

with $r \in (0,1)$. Invoking Theorem 3.3., we arrive at the bound

$$|f(x) - s_{f^*_\omega,X}(x)| \leq o(h^{rk}) + c'_n h^{r(q-d/2)+n(1-r)}$$

with $q = \min(m,k)$. By choosing $n \in \mathbb{N}$ such that $n > r(k-q+d/2)/(1-r)$, we can get the required result. □

Now, in what follows, we consider the well-known smooth conditionally positive definite radial functions. In this case, in order to get sharper error bounds, we employ the results in [10].

Example 4.1 Let the radial basis function ϕ be chosen to be one of what follows:

(a) $\phi(x) := (-1)^{\lceil m-d/2 \rceil}(|x|^2 + \lambda^2)^{m-d/2}$, d odd, $m > d/2$, (multiquadrics),

(b) $\phi(x) := (-1)^{m-d/2+1}(|x|^2 + \lambda^2)^{m-d/2} \log(|x|^2 + \lambda^2)^{1/2}$, $m > d/2$, d even, ('shifted' surface splines),

(c) $\phi(x) := (|x|^2 + \lambda^2)^{m-d/2}$, $0 < m < d/2$, (inverse multiquadrics),

where d, $m \in \mathbb{N}$ and $\lambda > 0$, and where $\lceil s \rceil$ indicates the smallest integer greater than s. We find (see [GS]) that the Fourier transform of ϕ is of the form

$$\hat{\phi} = c(m,d)\tilde{K}_m(\lambda \cdot)|\cdot|^{-2m}$$

where $c(m,d)$ is a positive constant depending on m and d, and $\tilde{K}_\nu(|t|) := |t|^\nu K_\nu(|t|) \neq 0$, $t \geq 0$, with $K_\nu(|t|)$ the modified Bessel function of order ν. It is well-known from literature (e.g., [AS]) that $\tilde{K}_\nu \sim (1+|\cdot|^{(2\nu-1)/2})\exp(-|\cdot|)$.

Corollary 4.3. *Let ϕ be one of the basis functions in the above Example 4.1. Let $s_{f^*_\omega,X}$ in (2.2) be an interpolant to f^*_ω on X by way of employing ϕ_ω. Let $\omega(h) = h|\log h|^r$ with $r > 1$. Then, for every $f \in W^k_\infty(\mathbb{R}^d) \cap W^q_2(\mathbb{R}^d)$ with $q = \min(k,m)$, we have the error bound*

$$|f(x) - s_{f^*_\omega,X}(x)| = o(h^k|\log h|^{rk}), \quad x \in \Omega.$$

Proof. Due to the result of Madych and Nelson (see [10]), we can find constants c_1, $c_2 > 0$ independent of X and Ω such that

$$\begin{aligned} P_{\phi,X/\omega}(x/\omega) &\leq c_1 \exp(-c_2|\log h|^r) \\ &\leq c_1 h^{c_2|\log h|^{r-1}}, \end{aligned}$$

where $\omega(h) = h|\log h|^r$ with $r > 1$. Then, for any given $n \in \mathbb{N}$, there exists a number $h_n > 0$ such that for any $h < h_n$, we get $c_2|\log h|^{r-1} > n$. It follows that

$$h^{c_2|\log h|^{r-1}} \leq c_1 h^n.$$

Thus, this corollary is proved immediately by Theorem 3.3. $\square$

Example 4.2. Let us consider the basis function ϕ whose Fourier transform $\hat{\phi}$ is of the form

$$\hat{\phi}(\theta) = \exp(-\alpha|\theta|^a), \quad \alpha > 0$$

with $0 < a \le 2$. In the case $a = 1$ and $\alpha = 1$, the function ϕ becomes the Poisson kernel

$$\phi = \frac{c_d}{(1 + |\cdot|^2)^{(d+1)/2}}$$

with a suitable constant c_d. When $a = 2$, it is the Gaussian function. Then, by applying the same technique in the above corollary, we can get the following result:

Corollary 4.4. *Let ϕ be one of the basis functions in Example 4.2. Let $s_{f^*_\omega,X}$ in* (2.2) *be an interpolant to f^*_ω on X by way of employing ϕ_ω, where $\omega(h) = h|\log h|^r$ with $r > 1$. Then, for every $f \in W^k_\infty(\mathbb{R}^d) \cap W^m_2(\mathbb{R}^d)$, we have the error bound*

$$|f(x) - s_{f^*_\omega,X}(x)| = o(h^k|\log h|^{rk}), \quad x \in \Omega.$$

Acknowledgments. This work was supported by the grants DoD-N00014-97-1-0806 and DEPSCoR-DAAG55-98-1-0002.

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