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Convergence analysis for a second-order elliptic equation by a collocation method using scattered points

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Abstract

A collocation method using scattered points applied to a second-order elliptic differential equation is analyzed by establishing a new quadrature formula for the space of the polynomials. We show that a polynomial solution possesses stability and preserves a similar convergence property occurred in the classical high order collocation method.

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1. Introduction

Consider the following model problem:

$$Lu := -[u_{xx} + u_{yy}] + cu = f, \quad \text{in } \Omega \equiv (-1, 1) \times (-1, 1), \tag{1.1}$$

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with homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \ \partial\Omega,$$

where c is a positive constant. The traditional high-order collocation method for solving (1.1) is to approximate u by a polynomial at Legendre–Gauss (:=LG)-type or Chebyshev–Gauss (:=CG) type points (see [2,3,8,10], etc.). The usages of LG points or CG points in the classical collocation method for (1.1) yield the so-called spectral convergences and well-posed algebraic linear systems. It is well known that these approaches are very popular and accurate to approximate the solutions of boundary value problems like (1.1) among many other numerical techniques. Despite such merits on the classical high-order collocation method, we are interested in a question on a spectral convergence property when arbitrarily chosen scattered points are used for collocating a given differential equation. Let $\mathscr{A}(u, v)$ be a bilinear form corresponding to (1.1) on a Sobolev space $H_0^1(\Omega)$ and $\mathscr{F}(v)$ a linear functional on $H_0^1(\Omega)$ such as

$$\mathscr{A}(u, v) = \int_\Omega \nabla u \cdot \nabla v + 2\alpha u v \, \mathrm{d}\Omega, \quad \mathscr{F}(v) = \int_\Omega f v \, \mathrm{d}\Omega. \tag{1.2}$$

Then the corresponding variational formulation is to find $u \in H_0^1(\Omega)$ satisfying $\mathscr{A}(u, v) = \mathscr{F}(v)$ for all $v \in H_0^1(\Omega)$ and its Galerkin approximation is to find $u_N \in \mathscr{P}_N^0(\Omega)$ such that $\mathscr{A}_N(u_N, v) = \mathscr{F}_N(v)$ for all $v \in \mathscr{P}_N^0(\Omega)$, where $\mathscr{A}_N$ and $\mathscr{F}_N$ are discrete linear functionals of $\mathscr{A}$ and $\mathscr{F}$, respectively, and $\mathscr{P}_N^0$ is the space of polynomials vanishing on the boundary $\partial\Omega$. Then the convergence analysis can be done usually by estimating both the errors $\mathscr{A}(u, v) - \mathscr{A}_N(u, v)$, which depends on a numerical quadrature rule, and $\mathscr{F}(v) - \mathscr{F}_N(v)$, which depends on an interpolation operator defined at LG- or CG-type collocation points. In this paper, instead of using an interpolation operator for $\mathscr{F}_N(v)$, we will employ a projection operator for $\mathscr{F}_N(v)$ as an approximation to $\mathscr{F}(v)$. This is because the interpolation error estimates are not known yet at general collocation points. We will use a new quadrature formula to approximate $\mathscr{A}(u, v)$ so that $\mathscr{A}(u, v) - \mathscr{A}_N(u, v)$ vanishes on a polynomial space. In some of the literatures there have been some efforts to exploit arbitrarily scattered points as collocation points: for example, the differentiation matrix for unstructured grids is introduced and applied to hyperbolic equations in [5]. A collocation method using a radial basis function is reported in [6] for solving Poisson's (1.1) numerically, in which nearly complete geometric freedom is allowed. Some developments of spectral methods on triangles and tetrahedra are also reported in [9], in which the approximate solutions can be represented as multivariate Lagrange interpolation polynomials. In this paper, we choose any scattered points and then approximate the solution u of the Eq. (1.1) by a polynomial at chosen scattered points in Ω. For these purposes, let $\mathscr{X} = \{s_j : j = 0, \ldots, M\}$ be a finite set of distinct points in $I = [-1, 1]$ and let $\mathscr{P}_N(I)$ be the set of polynomials of degree less than or equal to N.

For a given set of scattered points in Ω, we establish a quadrature formula on Ω, which is exact for polynomials of degree $\leqslant K$ by using $(M+1)$ scattered points with $(M+1) > \dim \mathscr{P}_K(I)$. The key point in this numerical quadrature is to find a unique kernel $\mathscr{A}(t) := (a(t, s_l))_{l=0}^M$ such that for all $p \in \mathscr{P}_K(I)$,

$$\sum_{l=0}^{M} p(s_l) a(t, s_l) = p(t). \tag{1.3}$$

Indeed, there are infinitely many solutions $a(\cdot, s_i)$ of the above linear system (1.3). However, it can be uniquely determined by making the kernel $a(t, s_l)$ in (1.3) be defined locally (we will see in Section 2).

For the weights in the case of two or higher dimensions, the tensor arguments will be used. Then we consider the corresponding discrete variational collocation formulation on a polynomial space, that is, we want to find a polynomial solution $u_N \in \mathscr{P}_N^0(\Omega)$ satisfying

$$\mathscr{A}_M(u_N, v_N) = [\Pi_N f, v_N]_M, \quad \forall v_N \in \mathscr{P}_N^0(\Omega), \tag{1.4}$$

where Π_N is a projection operator from a space of all continuous functions to $\mathscr{P}_N^0$, $\mathscr{A}_M(\cdot,\cdot)$ and $[\cdot,\cdot]_M$ which are defined in Section 3. Then we show that (1.4) has a unique solution. The spectral convergence analysis is also provided. The approach proposed here has the main advantage of providing spectral accuracies similar to the classical pseudo-spectral method with any scattered points in Ω. The standard Sobolev spaces $H^s(\Omega)$ and norms $\|u\|_s$ are used. For example, $L^2(\Omega)$ is same as $H^0(\Omega)$ whose inner product is given by (u, u) and its corresponding L^2 norm is given by $\|u\|$. The subspace $H_0^1(\Omega)$ of H^1 is the closure of C_0^∞. The seminorm $|u|_1$ is also used.

This paper is as follows: in Section 2, we derive a quadrature rule on Ω for any scattered points and then shows the way to get corresponding weights for one dimensional case. In Section 3, the variational collocation method and its related linear system are introduced. In Section 4, the convergence analysis is derived in the sense of L^2 and H^1 norms. In Section 5, we provide numerical tests for weights obtained in Section 2 and for L^2 and H^1 convergences of a polynomial solution. Finally, we add some concluding remarks in Section 6.

2. Numerical quadrature

Let $\mathscr{X} = \{s_j : j = 0, \ldots, M\}$ be a set of distinct points in I with $M+1 > K+1 := \dim(\mathscr{P}_K(I))$. Let $\mathscr{X}_j$ be the subset of $\mathscr{X}$ which has $(K+1)$-elements as follows:

$$\mathscr{X}_j = \begin{cases} \{s_l \in \mathscr{X} | l = 0, \ldots, K\}, & \text{if } 1 \leqslant j < \left\lfloor \dfrac{K+1}{2} \right\rfloor, \\ \left\{ s_l \in \mathscr{X} | l = j - \left\lfloor \dfrac{K+1}{2} \right\rfloor, \ldots, j + \left\lfloor \dfrac{K}{2} \right\rfloor \right\} & \text{if } \left\lfloor \dfrac{K+1}{2} \right\rfloor \leqslant j \leqslant M - \left\lfloor \dfrac{K}{2} \right\rfloor, \\ \{s_l \in \mathscr{X} | l = M-K, \ldots, M\} & \text{if } j > M - \left\lfloor \dfrac{K}{2} \right\rfloor. \end{cases} \tag{2.1}$$

Since the order of $\mathscr{P}_K$ is $K+1$, we can represent the polynomial $p \in \mathscr{P}_K$ as a linear combination of the $(K+1)$-number of values $p(s_l)$ for $s_l \in \mathscr{X}_j$, and corresponding polynomial $a_{j,l}(t)$ on the subinterval $[s_{j-1}, s_j)$ as follows:

$$\sum_{s_l \in \mathscr{X}_j} a_{j,l}(t) p(s_l) = p(t) \quad \text{for all } t \in [s_{j-1}, s_j). \tag{2.2}$$

Note that the kernel $a_{j,l}(t)$ in (2.2) is defined locally and it becomes a polynomial of degree K on $[s_{j-1}, s_j)$. Moreover, $a_{j,l}(t)$ is the Lagrange polynomial defined on $[s_{j-1}, s_j)$ such that

$$a_{j,l}(t) = \prod_{s_n \in \mathscr{X}_j, s_n \neq s_l} \frac{t - s_n}{s_l - s_n}, \quad \text{where} \quad t \in [s_{j-1}, s_j) \text{ and } s_l \in \mathscr{X}_j. \tag{2.3}$$

Therefore, the kernel $\mathscr{A}(t) = (a_{j,l}(t))_{l=0}^M$ becomes continuous piecewise polynomials.

Let

$$a(t, s_l) := \begin{cases} a_{j,l}(t), & t \in [s_{j-1}, s_j) \quad \text{if} \quad s_l \in \mathscr{X}_j, \\ 0, & t \in [s_{j-1}, s_j) \quad \text{if} \quad s_l \notin \mathscr{X}_j, \end{cases} \tag{2.4}$$

then it satisfies the relation

$$\sum_{l=0}^{M} p(s_l)a(t, s_l) = p(t), \quad \forall t \in [-1, 1], \ \forall p \in \mathscr{P}_K(I).$$

In fact, there is no restriction for the choice of the set X_j but the distribution of X_j is important in the sense of calculating the numerical weights $\hat{w}_j$ defined in (2.6) or (2.7). From the practical point of view, a good choice for approximation is to put almost the same numbers of centers on each side of the interval $[s_{j-1}, s_j)$. It is also interest to point out that an error bound usually depends on the maximal distance between centers.

Proposition 2.1. *Let* $\mathscr{X} := \{s_k : k = 0, 1, \ldots, M\}$ *with* $M > K$ *be a set of arbitrary distinct points in* $[-1, 1]$. *Assume that* (2.2) *holds. Then, for all polynomials* $p \in \mathscr{P}_K([-1, 1])$ *and a given function* $w(t)$ *defined on* $[-1, 1]$, *we have*

$$\int_{-1}^{1} p(t)w(t)\,\mathrm{d}t = \sum_{l=0}^{M} p(s_l)\hat{w}_l, \tag{2.5}$$

where $\hat{w}_l$ *is defined in* (2.6) *or* (2.7).

Proof. From (2.2) and (2.4), one may have

$$\begin{aligned}
\int_{-1}^{1} p(t)w(t)\,\mathrm{d}t &= \sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} p(t)w(t)\,\mathrm{d}t = \sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} \sum_{s_l \in \mathscr{X}_j} a_{j,l}(t)p(s_l)w(t)\,\mathrm{d}t \\
&= \sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} \sum_{l=0}^{M} p(s_l)a(t, s_l)w(t)\,\mathrm{d}t \\
&= \sum_{l=0}^{M} p(s_l) \sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} a(t, s_l)w(t)\,\mathrm{d}t \\
&= \sum_{l=0}^{M} p(s_l)\hat{w}_l,
\end{aligned}$$

where because of (2.1) and (2.4), the quadrature weights can be represented as follows: For the case of $K < M/2$, i.e. $K < M - K$,

$$\hat{w}_l = \begin{cases} \displaystyle\sum_{j=1}^{l+\lfloor\frac{K+1}{2}\rfloor} \int_{s_{j-1}}^{s_j} a_{j,l}(t)w(t)\,\mathrm{d}t & \text{for } l = 0, 1, \ldots, K, \\ \displaystyle\sum_{j=l-\lfloor\frac{K}{2}\rfloor}^{l+\lfloor\frac{K+1}{2}\rfloor} \int_{s_{j-1}}^{s_j} a_{j,l}(t)w(t)\,\mathrm{d}t & \text{for } l = K+1, \ldots, M-K-1, \\ \displaystyle\sum_{j=l-\lfloor\frac{K}{2}\rfloor}^{M} \int_{s_{j-1}}^{s_j} a_{j,l}(t)w(t)\,\mathrm{d}t & \text{for } l = M-K, \ldots, M. \end{cases} \tag{2.6}$$

For the case of $K \geqslant M/2$, i.e. $K \geqslant M - K$,

$$\hat{w}_l = \begin{cases} \displaystyle\sum_{j=1}^{l+\lfloor\frac{K+1}{2}\rfloor} \int_{s_{j-1}}^{s_j} a_{j,l}(t)w(t)\,\mathrm{d}t & \text{for } l = 0, 1, \ldots, M-K-1, \\ \displaystyle\sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} a_{j,l}(t)w(t)\,\mathrm{d}t & \text{for } l = M-K, \ldots, K, \\ \displaystyle\sum_{j=l-\lfloor\frac{K}{2}\rfloor}^{M} \int_{s_{j-1}}^{s_j} a_{j,l}(t)w(t)\,\mathrm{d}t & \text{for } l = K+1, \ldots, M. \end{cases} \tag{2.7}$$

This completes the proof. □

The above proposition tells us the way to get the numerical quadrature rule which is exact for a polynomial of degree up to K. We note that one may use a tensor argument to extend one-dimensional numerical quadrature formula to a higher-dimensional case immediately. In fact, one may notice that several numerical tests for the convergence for a two-dimensional case, which needs two-dimensional weights.

Now the weights $\hat{w}_j$ defined in (2.6) and (2.7) can be written explicitly with help of Legendre–Gauss-Lobatto [= :LGL] quadrature rule applied to the kernel $a_{j,l}(t)$ which is a polynomial of the degree K on the subinterval $[s_{j-1}, s_j]$ for $j = 1, 2, \ldots, M$ with $s_0 = -1$ and $s_M = 1$. For this, let $L_n(x)$ be the Legendre polynomial of degree n. Then the LGL quadrature points $\{\xi_j\}_{j=0}^n$ are the zeros of the polynomial $(1 - x^2)L_n'(x)$ a nd the corresponding LGL weights $\{\rho_j\}_{j=0}^n$ are

$$\rho_j = \frac{2}{n(1+n)[L_n(\xi_j)]^2}, \quad j = 0, 1, \ldots, n. \tag{2.8}$$

Then LGL quadrature rule is exact for polynomials of degree up to $2n$.

Now, we can represent the weights $\hat{w}_l$ in (2.6) and (2.7) for the case $w(t) = 1$ by using the quadrature form. Using the change of variables and the exactness of the LGL quadrature with $n = \lfloor K/2 \rfloor + 1$, the integral in the weights can be rewritten as

$$\begin{aligned} w(j,l) &:= \int_{s_{j-1}}^{s_j} a_{j,l}(t)\,\mathrm{d}t \\ &= \frac{s_j - s_{j-1}}{2}\int_{-1}^{1} \hat{a}_{j,l}(x)\,\mathrm{d}x = \frac{s_j - s_{j-1}}{2}\sum_{i=0}^{\lfloor \frac{K}{2}\rfloor+1} \hat{a}_{j,l}(\xi_i)\rho_i, \end{aligned} \tag{2.9}$$

where

$$\hat{a}_{j,l}(x) := a_{j,l}\left(\frac{(s_j - s_{j-1})x + (s_{j-1}+s_j)}{2}\right), \quad x \in [-1,1]. \tag{2.10}$$

Since the expression (2.3) of $a_{j,l}(t)$ leads to

$$w(j,l) = \frac{s_j - s_{j-1}}{2}\sum_{i=0}^{\lfloor \frac{K}{2}\rfloor+1} \rho_i \left[\prod_{s_n\in\mathscr{X}_j, s_n\neq s_l} \frac{s_{j-1}(1-\xi_i)+s_j(1+\xi_i)-2s_n}{2(s_l - s_n)}\right], \tag{2.11}$$

the weights $\hat{w}_l$ in (2.6) and (2.7) can be written as follows. For the case of $K < M/2$, i.e. $K < M - K$,

$$\hat{w}_l = \begin{cases} \displaystyle\sum_{j=1}^{l+\lfloor \frac{K+1}{2}\rfloor} w(j,l) & \text{for } l = 0,1,\ldots,K, \\ \displaystyle\sum_{j=l-\lfloor \frac{K}{2}\rfloor}^{l+\lfloor \frac{K+1}{2}\rfloor} w(j,l) & \text{for } l = K+1,\ldots,M-K-1, \\ \displaystyle\sum_{j=l-\lfloor \frac{K}{2}\rfloor}^{M} w(j,l) & \text{for } l = M-K,\ldots,M. \end{cases} \tag{2.12}$$

For the case of $K \geqslant M/2$, i.e. $K \geqslant M-K$,

$$\hat{w}_l = \begin{cases} \displaystyle\sum_{j=1}^{l+\lfloor \frac{K+1}{2}\rfloor} w(j,l) & \text{for } l = 0,1,\ldots,M-K-1, \\ \displaystyle\sum_{j=1}^{M} w(j,l) & \text{for } l = M-K,\ldots,K, \\ \displaystyle\sum_{j=l-\lfloor \frac{K}{2}\rfloor}^{M} w(j,l) & \text{for } l = K+1,\ldots,M. \end{cases} \tag{2.13}$$

Finally, one may note that the quadrature weights $\hat{w}_l$ can be determined in a different way. For example, since

$$\int_{-1}^{1} p(t)w(t)\,\mathrm{d}t = \int_{-1}^{1}\sum_{l=0}^{M} p(s_l)a(t,s_l)w(t)\,\mathrm{d}t = \sum_{l=0}^{M} p(s_l)\sum_{j=0}^{\lfloor \frac{K}{2}\rfloor+1} a(\xi_j, s_l)\rho_j,$$

the quadrature weights can be given as $\hat{w}_l = \sum_{j=0}^{\lfloor \frac{K}{2}\rfloor+1} a(\xi_j, s_l)\rho_j$.

3. A variational collocation method

From now on we assume that the weight function $w(x)=1$. Let $\mathscr{X}=\{s_i \mid i=0,1,\dots,M\}$ where $M>2N$. Then the numerical quadrature in the previous section shows that

$$\int_{-1}^{1} p(t)\,\mathrm{d}t = \sum_{i=0}^{M} p(s_i)\hat{w}_i \quad \text{for all} \quad p \in \mathscr{P}_{2N}(I). \tag{3.1}$$

3.1. One-dimensional case

In this section, we describe one-dimensional high-order variational collocation method using scattered points and a projection operator for the following model problem:

$$-u'' + cu = f \quad \text{in } (-1,1), \quad u(-1)=u(1)=0, \tag{3.2}$$

where c is a positive constant.

The variational form for (3.2) is to find $u \in H_0^1(I)$ such that

$$\mathscr{A}(u,v) = \mathscr{F}(v) \quad \text{for all} \quad v \in H_0^1(I), \tag{3.3}$$

where the bilinear functional $\mathscr{A}(\cdot,\cdot) : H_0^1(I) \times H_0^1(I) \longrightarrow \Re$ and the linear functional $\mathscr{F}(\cdot) : C[-1,1] \longrightarrow \Re$ are defined as

$$\mathscr{A}(u,v) := \int_I u'v' + cuv\,\mathrm{d}t, \quad \mathscr{F}(v) = \int_I fv\,\mathrm{d}t. \tag{3.4}$$

Assume that there is an orthogonal projection operator $\pi_N : L^2(I) \longrightarrow \mathscr{P}_N(I)$ with the following property: for all $f \in L^2(I)$,

$$\|f - \pi_N f\| \leqslant CN^{-m}\|f\|_m, \tag{3.5}$$

where $m \geqslant 1$ is an integer (see [2]). The variational collocation formulation corresponding to (3.3) is to find $u_N \in \mathscr{P}_N^0(\Omega)$ such that

$$\mathscr{A}_M(u_N, v_N) = \mathscr{F}_M(v_N) \quad \text{for all} \quad v_N \in \mathscr{P}_N^0(I), \tag{3.6}$$

where the discrete bilinear functional $\mathscr{A}_M(\cdot,\cdot) : \mathscr{P}_N^0(I) \times \mathscr{P}_N^0(I) \longrightarrow \Re$ and the discrete linear functional $\mathscr{F}_M : \mathscr{P}_N^0(I) \longrightarrow \Re$, corresponding to $\mathscr{A}(\cdot,\cdot)$ and $\mathscr{F}(\cdot)$, respectively, are defined as

$$\mathscr{A}_M(u_N, v_N) = \sum_{i=0}^{M} [u_N'(s_i)v_N'(s_i) + cu_N(s_i)v_N(s_i)]\hat{w}_i \tag{3.7}$$

and

$$\mathscr{F}(v_N) = \sum_{i=0}^{M} (\pi_N f)(s_i)v_N(s_i)\hat{w}_i. \tag{3.8}$$

Due to the exactness of numerical quadrature, we note that

$$\mathscr{A}_M(u_N, v_N) = \int_I u_N' v_N' + c u_N v_N \,\mathrm{d}t = \int_I (-u_N'' + c u_N) v_N \,\mathrm{d}t$$
$$= \sum_{i=0}^{M} (-u_N'' + c u_N)(s_i) v_N(s_i) \hat{w}_i. \tag{3.9}$$

Let

$$u_N(t) = \sum_{j=0}^{N-2} u_j p_j(t) \in \mathscr{P}_N^0(I),$$

where $\{p_j(t)\}$ is a basis for $\mathscr{P}_N^0(I)$. Then we can define the matrices B and E whose elements are given as

$$B(i,k) = -p_k''(s_i), \quad E(i,k) = p_k(s_i) \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant k \leqslant N-2.$$

Note that

$$(Lp_k)(s_i) = -p_k''(s_i) + c p_k(s_i) \tag{3.10}$$

leads to the $(M+1) \times (N-1)$ matrix

$$(B + cE) =: \hat{B}. \tag{3.11}$$

Then the matrix (3.11) results in an over-determined system which takes after the usual collocation method for the approximations of the given differential (3.2). But there is no solution usually for such an over-determined system. The matrix representation of (3.6) becomes

$$E^{\mathrm{T}} \hat{W} \hat{B}\, U_N = E^{\mathrm{T}} \hat{W} F_M, \tag{3.12}$$

where

$$\hat{W} = diag(\hat{w}_j)$$

and, with vectors arranged by the same order as basis,

$$U_N = (u_0, \ldots, u_{N-2})^{\mathrm{T}} \quad \text{and} \quad F_M = ((\pi_N) f(s_0), \ldots, (\pi_N f)(s_M))^{\mathrm{T}}.$$

We note that $E^T \hat{W} \hat{B}$ is the square matrix with the size $N-1$ because of zero boundary conditions, which enables us to solve the linear system (3.12) if it has an inverse. The matrix $E^{\mathrm{T}} \hat{W} \hat{B}$ is not symmetric positive definite, but we can obtain the symmetric positive definite system directly from (3.7):

$$(D^{\mathrm{T}} \hat{W} D + c E^{\mathrm{T}} \hat{W} E) U_N = E^{\mathrm{T}} \hat{W} F_M, \tag{3.13}$$

where $D(i,k) = p_k'(s_i)$ with $k = 0, 1, \ldots, N-2$ and $i = 0, 1, \ldots, M$. This matrix (3.13) which comes from the variational collocation problem (3.6) will be used for actual computations in Section 5.

3.2. Two-dimensional case

Consider the two-dimensional model problem (1.1)

$$Lu := -[u_{xx} + u_{yy}] + cu = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Assume that there is $L^2(\Omega)$ orthogonal projection operator $\Pi_N : L^2(\Omega) \longrightarrow \mathscr{P}_N(\Omega)$ such that for all function $f \in L^2(\Omega)$, $\Pi_N f \in \mathscr{P}_N(\Omega)$ which has the following properties: for all $f \in L^2(\Omega)$

$$\|f - \Pi_N f\| \leqslant CN^{-m}\|f\|_m \tag{3.14}$$

and

$$\|f - \Pi_N f\|_1 \leqslant CN^{\frac{3}{2}-m}\|f\|_m, \tag{3.15}$$

where $m \geqslant 1$ is an integer and the constant C does not depend on N (see [10]). Let $\mathscr{X}^2 = \mathscr{X} \times \mathscr{X}$ be the set of scattered points in Ω. The variational form for (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$\mathscr{A}(u, v) = \mathscr{F}(v) \quad \text{for all} \quad v \in H_0^1(\Omega), \tag{3.16}$$

where the bilinear functional $\mathscr{A}(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \Re$ and the linear functional $\mathscr{F}(\cdot) :\longrightarrow \Re$ are defined as

$$\mathscr{A}(u, v) := \int_\Omega \nabla u \cdot \nabla v + cuv \, d\Omega, \quad \mathscr{F}(v) = \int_\Omega fv \, d\Omega. \tag{3.17}$$

Now consider the variational collocation formulation corresponding to (3.16): find $u_N \in \mathscr{P}_N^0(\Omega)$ such that

$$\mathscr{A}_M(u_N, v_N) = [\Pi_N f, v_N]_M \quad \text{for all} \quad v_N \in \mathscr{P}_N^0(\Omega), \tag{3.18}$$

where the discrete bilinear functional $\mathscr{A}_M(\cdot,\cdot) : \mathscr{P}_N^0(\Omega) \times \mathscr{P}_N^0(\Omega) \longrightarrow \Re$ and the discrete linear functional $\mathscr{F}_M : \mathscr{P}_N^0(\Omega) \longrightarrow \Re$, corresponding to $\mathscr{A}_M(\cdot,\cdot)$ and $\mathscr{F}_M(\cdot)$, respectively, are defined as

$$\mathscr{A}_M(u_N, v_N) = \sum_{i,j=0}^{M} [\nabla u_N(s_i, s_j) \cdot \nabla v_N(s_i, s_j) + cu_N(s_i, s_j)v(s_i, s_j)]w_i^x w_j^y \tag{3.19}$$

and

$$\mathscr{F}_M(v_N) = \sum_{i,j=0}^{M} (\Pi_N f)(s_i, s_j)v_N(s_i, s_j)\hat{w}_i^x \hat{w}_j^y. \tag{3.20}$$

The two-dimensional scattered points of $\overline{\Omega}$ can be arranged as

$$\hat{s}_\mu = (s_i, s_j), \quad \mu = j + i(M+1), \quad i, j = 0, \ldots, M$$

and, accordingly, the polynomial basis for $P_N^0(\Omega)$ can be also arranged as

$$p_v(x, y) = p_k(x)p_n(y), \quad v = n + k(N-1), \quad k, n = 0, \ldots, N-2.$$

Let

$$u_N(x,y) = \sum_{\nu=0}^{(N-1)^2-1} u_\nu p_\nu(x,y) \in \mathscr{P}_N^0(\Omega).$$

Define the two collocation matrices D and E, respectively, as

$$\begin{aligned} E(i,j) &:= p_j(s_i) = (1-s_i^2)L_j(s_i), \\ D(i,j) &:= p'_j(s_i) = (1-s_i^2)L'_j(s_i) - 2s_i L_j(s_i), \quad i=0,1,\ldots,M, \quad j=0,1,\ldots,N-2, \end{aligned}$$

where $L_j(t)$ is the Legendre polynomial. Then the matrix versions of (3.19) and (3.20) can be written as

$$(\hat{D}_x^{\mathrm{T}}\hat{W}\hat{D}_x + \hat{D}_y^{\mathrm{T}}\hat{W}\hat{D}_y + c\,\hat{E}^{\mathrm{T}}\hat{W}\hat{E})U_N = \hat{E}^{\mathrm{T}}\hat{W}F_M, \tag{3.21}$$

where

$$\hat{D}_x = D \otimes E, \quad \hat{D}_y = E \otimes D, \quad \hat{E} = E \otimes E, \quad \hat{W} = W^x \otimes W^y$$

and

$$U_N = (u_0, \ldots, u_{(N-1)^2-1})^{\mathrm{T}} \quad \text{and} \quad F_M = ((\Pi_N)f(s_0), \ldots, (\Pi_N f)(s_{(M+1)^2-1}))^{\mathrm{T}}$$

with vectors arranged by the same order as basis.

4. Convergence analysis

In this section, we discuss the uniqueness, stability and convergence for the problem (3.18):

Proposition 4.1. *The problem* (3.18) *has a unique solution* $u_N \in \mathscr{P}_N^0(\Omega)$. *Moreover, the unique solution* $u_N \in \mathscr{P}_N^0(\Omega)$ *satisfies that for all continuous function* f *on* Ω,

$$\|u_N\|_1 \leqslant C\|\Pi_N f\|, \tag{4.1}$$

where the constant C *is independent of* N.

Proof. Because of the exactness of numerical quadrature and Green's formula, $\mathscr{A}_M(\cdot,\cdot)$ satisfies for some positive constants C_1 and C_2

$$\mathscr{A}_M(u_N,u_N) = \int_\Omega \nabla u_N \cdot \nabla u_N + c u_N u_N \,\mathrm{d}\Omega \geqslant C_1\|u_N\|_1^2 \tag{4.2}$$

and

$$\mathscr{A}_M(u_N,v_N) = \int_\Omega \nabla u_N \cdot \nabla v_N + c\,u_N v_N \,\mathrm{d}\Omega \leqslant C_2\|u_N\|_1\,\|v_N\|_1. \tag{4.3}$$

Therefore, the Lax–Milgram lemma yields the existence and the uniqueness of the solution of (3.18). Using Cauchy–Schwarz inequality, we have for $v_N \in \mathscr{P}_N^0(\Omega)$

$$\begin{aligned}\mathscr{F}_M(u_N) &= \sum_{i,j=0}^{M} (\Pi_N f)(s_i, s_j) u_N(s_i, s_j) \hat{w}_i^x \hat{w}_j^y \\ &\leqslant \left(\sum_{i,j=0}^{M} (\Pi_N f)(s_i, s_j)^2 \hat{w}_i^x \hat{w}_j^y \right)^{1/2} \left(\sum_{i,j=0}^{M} u_N(s_i, s_j)^2 \hat{w}_i^x \hat{w}_j^y \right)^{1/2} \\ &= \left(\int_\Omega |\Pi_N f|^2 \right)^{1/2} \left(\int_\Omega |u_N(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \\ &= \|\Pi_N f\| \, \|u_N\|_1. \end{aligned} \tag{4.4}$$

The stability estimate (4.1) comes from (4.2) and (4.4). □

Because of Proposition 4.1, the problem (3.18) has the unique solution with a stability.

Theorem 4.1. *Let u be the solution of* (3.16) *and* u_N *be the solution of the problem* (3.18). *Then we have the following error estimate*:

$$\|u - u_N\|_1 \leqslant C \left(\inf_{v_N \in \mathscr{P}_N^0(\Omega)} \|u - v_N\|_1 + \|f - \Pi_N f\| \right), \tag{4.5}$$

where the constant C depends on Ω.

Proof. Due to (4.2) and (3.18), it follows that for any $v_N \in P_N^0(\Omega)$

$$C_1 \|u_N - v_N\|_1^2 \leqslant \mathscr{A}_M(u_N - v_N, u_N - v_N) = \mathscr{F}_M(u_N - v_N) - \mathscr{A}_M(v_N, u_N - v_N). \tag{4.6}$$

Let u be the solution of (3.16). Then, using (4.6), the exactness of numerical quadrature, Schwarz inequality and Poincare inequality yields

$$\begin{aligned} C_1 \|u_N - v_N\|_1^2 &= \mathscr{A}(u, u_N - v_N) - \mathscr{F}(u_N - v_N) + \mathscr{F}_M(u_N - v_N) - \mathscr{A}_M(v_N, u_N - v_N) \\ &= \mathscr{A}(u - v_N, u_N - v_N) - \mathscr{F}(u_N - v_N) + \mathscr{F}_M(u_N - v_N) \\ &\leqslant C(\|u - v_N\|_1 + \|f - \Pi_N f\|) \, \|u_N - v_N\|_1. \end{aligned} \tag{4.7}$$

Due to (4.7) and the triangle inequality, we can deduce that for any $v_N \in P_N^0(\Omega)$

$$\|u - u_N\|_1 \leqslant \|u - v_N\|_1 + \|v_N - u_N\|_1 \leqslant C(\|u - v_N\|_1 + \|f - \pi_N f\|). \tag{4.8}$$

Hence this argument yields the conclusion. □

Corollary 4.1. *Suppose that the solution u of the problem* (3.16) *is in* H^s *with an integer* $s \geqslant 1$ *and* $f \in H^r$ *with an integer* $r \geqslant 1$. *Then, for problem* (3.18), *we have*

$$\|u - u_N\|_1 \leqslant C(N^{(\frac{3}{2} - s)} \|u\|_s + N^{-r} \|f\|_r). \tag{4.9}$$

Proof. Since (4.5) holds for any $v_N \in P_N^0(\Omega)$, the result comes from (3.15) and (3.14). □

Theorem 4.2. *Under the hypothesis of Corollary* 4.1, *for problem* (3.18), *we have*

$$\|u-u_N\|\leqslant C(N^{1-s}\|u\|_s+N^{-r}\|f\|_r). \tag{4.10}$$

Proof. Letting $\mathbf{t}=(x,y)$, we note that

$$\|u-u_N\|=\sup_{g\in L^2(\Omega)}\frac{\int_\Omega (u-u_N)(\mathbf{t})g(\mathbf{t})\,\mathrm{d}\mathbf{t}}{\|g\|}. \tag{4.11}$$

The solution $w\in H_0^1(\Omega)$ to the problem

$$\mathscr{A}(w,v)=\int_\Omega g(\mathbf{t})v(\mathbf{t})\,\mathrm{d}\mathbf{t}\quad \forall v\in H_0^1(\Omega),\ \forall g\in L^2(\Omega), \tag{4.12}$$

satisfies

$$\|w\|_2\leqslant C\|g\|. \tag{4.13}$$

Using the exactness of numerical quadrature, (3.16) and (3.18), we have for all polynomial $w_N\in\mathscr{P}_N^0(\Omega)$

$$\mathscr{A}(u-u_N,w_N)=\mathscr{F}(w_N)-\mathscr{F}_M(w_N)=\int_\Omega (f-\Pi_N f)(\mathbf{t})w_N(\mathbf{t})\,\mathrm{d}\mathbf{t}$$

and, from (4.12), we have

$$\begin{aligned}\int_\Omega (u-u_N)(\mathbf{t})g(\mathbf{t})\,\mathrm{d}\mathbf{t}&=\mathscr{A}(u-u_N,w)\\&=\mathscr{A}(u-u_N,w-w_N)+\mathscr{A}(u-u_N,w_N)\\&=\mathscr{A}(u-u_N,w-w_N)+\int_\Omega (f-\Pi_N f)(\mathbf{t})w_N(\mathbf{t})\,\mathrm{d}\mathbf{t}.\end{aligned}$$

Now using the continuity of $\mathscr{A}(\cdot,\cdot)$, Schwarz inequality, Poincare inequality, (3.15) and (3.14) lead to

$$\begin{aligned}\int_\Omega (u-u_N)(\mathbf{t})g(\mathbf{t})\,\mathrm{d}\mathbf{t}&\leqslant C(|u-u_N|_1|w-w_N|_1+\|f-\Pi_N f\||w_N|_1)\\&\leqslant C(|u-u_N|_1|w-w_N|_1+\|f-\Pi_N f\|(|w-w_N|_1+\|w\|_1))\\&\leqslant C(N^{-\frac{1}{2}}|u-u_N|_1\|w\|_2+N^{-r}\|f\|_r(N^{-\frac{1}{2}}\|w\|_2+\|w\|_1))\\&\leqslant C(N^{-\frac{1}{2}}|u-u_N|_1+N^{-r}\|f\|_r)\|w\|_2.\end{aligned} \tag{4.14}$$

Combining (4.11), (4.9) and (4.13), we have the conclusion. □

5. Numerical example

Let $\Omega=(-1,1)\times(-1,1)$. Then the following model problem:

$$\begin{aligned}-[u_{xx}+u_{yy}]+u=(2\pi^2+1)\sin\pi x\ \sin\pi y\quad&\text{in}\quad \Omega,\\ u=0\quad&\text{on}\quad \partial\Omega,\end{aligned}$$

Table 1
Quadrature weights for equally spaced points on $[-1, 1] : s_i = -1 + 2i/M, \quad i = 0, 1, \ldots, M$

Scattered points selected (s_i)	($M = 100$, $K = 50$)	
	Quadrature weights	
	64 bit mantissa	128 bit mantissa
−1.000000	4.02917283811215760504e−03	4.0291728381121576050318324003355185138300e−03
−0.800000	−7.60851091955374213645e +05	−7.6085109195537421421258446573704958596220e + 05
−0.600000	−1.71855216754343152609e + 09	−1.7185521675434315270039482440802818856180e + 09
−0.400000	−1.13986966342148075378e + 09	−1.1398696634214807543287389736969204759000e + 09
−0.200000	−1.85861753522131415982e + 05	−1.8586175352213141603676075079582469210430e + 05
0.000000	1.99710943219752775376e−02	1.9971094321975277577306305565301168220880e−02
0.200000	−1.85861753522144483227e + 05	−1.8586175352214448329183946018702610958420e + 05
0.400000	−1.13986966342154070640e + 09	−1.1398696634215407068950431256920132873730e + 09
0.600000	−1.71855216754349147870e + 09	−1.7185521675434914795702523961333331945410e + 09
0.800000	−7.60851091955387280830e + 05	−7.6085109195538728146766383624373926695960e + 05
1.000000	4.02917283811215633295e−03	4.0291728381121563329425389529381701402660e−0
Error of weights	6.95614e−10	4.18957e−29

will be taken for a numerical evidence to support the convergence results. Furthermore, we provide here one-dimensional weights for several cases numerically. Note that the above model problem has the unique solution of $u(x, y) = \sin \pi x \sin \pi y$.

To obtain the quadrature weights $\hat{w}_l$, we need to evaluate Lagrange polynomials (2.3). But it is well known that this polynomial is very unstable for higher degree. This instability results in difficulties for achievement of our goals by usual single/double precision arithmetic for floating-point numbers, which have 24 bit mantissa and 53 bit mantissa, respectively: even if we work with double precision floating-point arithmetic, we cannot avoid significant digit cancellation in calculation of Lagrange polynomial of high degree because of roundoff error. To perform the computation of Lagrange polynomials of higher degree successfully, we adopt the 128 bits precision floating-point numbers in GNU MP (Gnu multiple precision library [11]), which gives an arbitrary precision. With the help of this library, we can obtain quadrature weights with high accuracy of about 30 decimal places (see Tables 1–3). In each table, we list some selected scattered points and their corresponding weights. The 64 and 128 bit mantissa have about 19 and 38 significant decimal digits respectively. The error of weights means the difference of 2 and the sum of all weights.

Now, we solve this system by a conjugate gradient method without preconditioner [1]. In Figs. 1 and 2, we report the L^2 and H^1 Sobolev norm for the Legendre variational collocation method. In these numerical tests, we choose $M = 50$ or 100 points in each x and y directions, so that 2500 or 10 000 points are taken in the computational domain. The degrees of polynomial solutions P_N are 5, 10, 15, 20 and 25. These numerical tests for the model problem show the spectral convergence for L^2 and H^1 sense, regardless of choice and numbers of scattered points. In the computational aspects, the scattered points distributed like Chebyshev points are preferred to other cases of scattered points.

Table 2
Quadrature weights for Chebyshev points: $s_i = -\cos(\pi i/M), \quad i = 0, 1, \ldots, M$

Selected scattered points selected (s_i)	($M = 100$, $K = 50$)	
	Quadrature weights	
	64 bit mantissa	128 bit mantissa
−1.000000	2.32935338587141686305e + 00	2.3293533858714168634259715344273372182400e + 00
−0.951057	8.00625486552649712763e−01	8.0062548655264971290341234331626468922780e−01
−0.809017	2.22791138702256323830e−02	2.2279113870225632416548269815690753093920e−02
−0.587785	2.54163839080124230381e−02	2.5416383908012423085239114249391203144620e−02
−0.309017	2.98783216475778899574e−02	2.9878321647577890014645834220479414502480e−02
−0.000000	3.14159265358979155886e−02	3.1415926535897915649859040146932568526040e−02
0.309017	2.98783216470020877059e−02	2.9878321647002087761833506600831470489130e−02
0.587785	2.54150954992010541524e−02	2.5415095499201054191521756175941039097470e−02
0.809017	8.98331715608820901899e−03	8.9833171560882090412431595850733820102600e−03
0.951057	−1.91107025541142409354e + 00	−1.9110702554114240944084441847962086023290e + 00
1.000000	−5.58115812046907569974e + 00	−5.5811581204690757010702171442217274793660e + 00
Error of weights	5.58364e−18	2.43915e−37

Table 3
Quadrature weights for uniformly distributed random on [−1, 1) and two boundary points {−1, 1}

Selected scattered points selected (s_i)	($M = 100$, $K = 50$)	
	Quadrature weights	
	64 bit mantissa	128 bit mantissa
−1.000000	1.55997919875841053460e−03	1.5599791987584105345999904328946737269940e−03
−0.845511	5.56799202172195958111e + 04	5.5679920217219595882541373137368545720560e + 04
−0.721106	−6.57600833873745034301e + 04	−6.5760083387374503711893779792991925149040e + 04
−0.562618	−1.39612620096338682462e + 07	−1.3961262009633868168739221225302553257090e + 07
−0.367481	1.21165794799104808568e−01	1.2116579479910480873212052492391232347310e−01
−0.155688	6.89830838967550559543e + 00	6.898308389675505598526085087745710717710e + 00
0.125146	−3.23140926649002962698e + 00	−3.2314092664900296298255051908814344688890e + 00
0.344232	−1.55234935071183041492e + 04	−1.5523493507118304155216200460561821952750e + 04
0.633373	−9.36761760876486325931e + 02	−9.3676176087648632647900555313720198862930e + 02
0.909896	−9.02921663242048124705e + 00	−9.0292166324204812583022040741136123959140e + 00
1.000000	3.12917387233774157955e−03	3.1291738723377415794837571923855141609520e−03
Error of weights	−3.03417e −11	−1.92780e−30

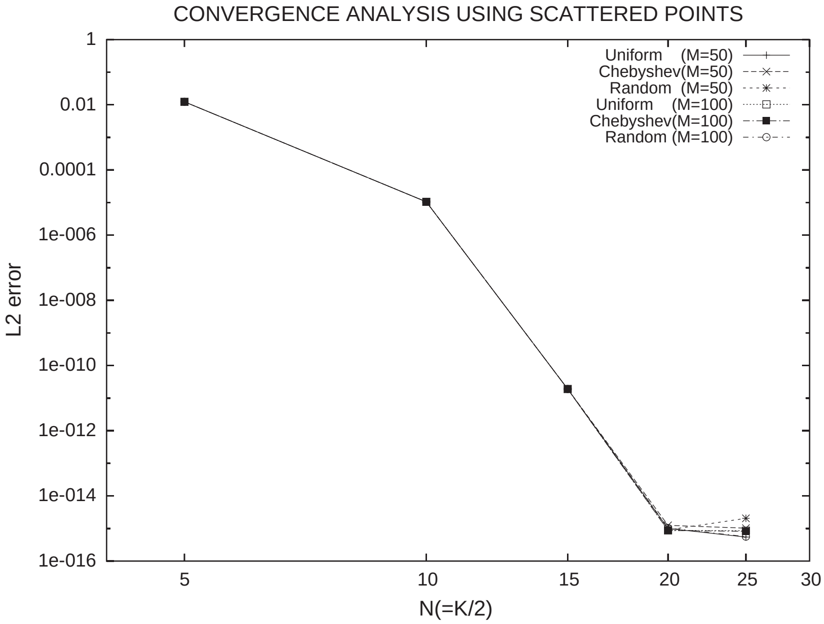

Fig. 1. L_2–error norms.

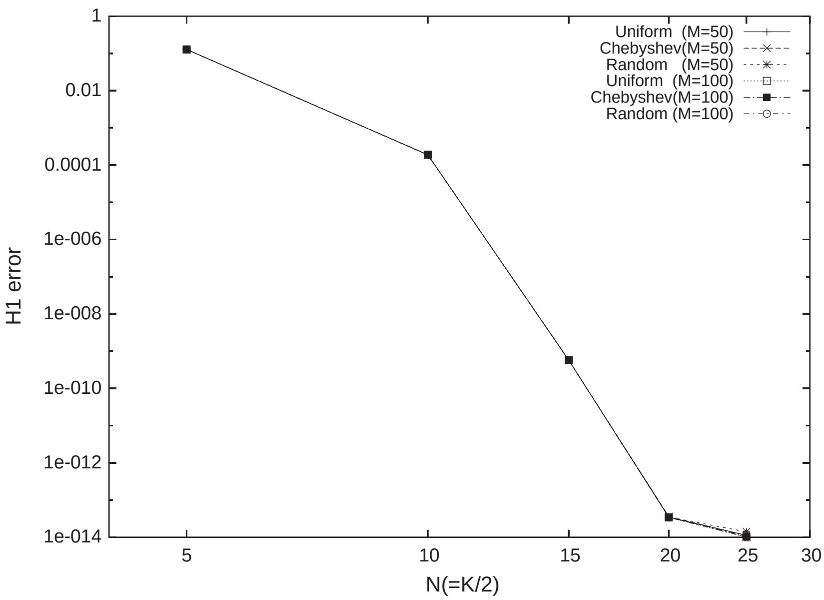

Fig. 2. H_1–error norms.

6. Remarks

The convergence and stability analysis depend on the developed numerical quadrature and projection operator. The numerical quadrature (2.12) or (2.13) relies on the choices of sets which consist of the arbitrarily $(K+1)$-number of the points. As known, the LG- or CG-type points and weights are computationally good enough for an approximation of a large class of boundary value differential equations. In this case, the given differential equation is collocated at such points and the interpolation operator will be used for the convergence and stability analysis (see [2,3,10], for example). In this paper, the points are designated arbitrarily for collocating the model problem (1.1) in the variational sense and the projection operator is used. For the convergence analysis, the known techniques are used (see [2,3,10], for example). Even though the convergence is shown analytically and numerically, it still remains how we can develop an efficient way for accurate computations of numerical solutions including numerical weights. These questions will be studied in a forthcoming paper. Finally, note that scattered data collocation problem has been also considered by using a radial basis function interpolation. It might be of interest to note that the radial basis function interpolant converges to the polynomial interpolant when the radial basis function is made increasingly flat (see [7]).

References

[1] R. Barrett, M. Berry, T.F. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, H. Van Der Vorst, Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods, SIAM, Philadelphia, PA, 1994.
[2] C. Bernardi, Y. Maday, Approximations Spectrales de Problèmes aux Limites Elliptiques, Springer, Berlin, 1992.
[3] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer, Berlin, 1987.
[5] M.H. Carpenter, D. Gottlieb, Spectral methods on arbitrary grids, J. Comput. Phys. 129 (1996) 74–86.
[6] B. Fornberg, E. Larsson, Solving elliptic PDEs using radial basis functions, ICOSAHOM-01 International Conference on Spectral and High Order Methods, June 11–15, Book of Abstract, Uppsala University, Sweden, 2001.
[7] B. Fornberg, G. Wright, E. Larsson, Some observations regarding interpolants in the limit of flat radial basis functions, Comput. Math. Appl. 47 (2004) 37–55.
[8] D. Funaro, Polynomial Approximation of Differential Equations, Lecture Notes in Physics, Springer, Berlin, 1992.
[9] J.S. Hesthaven, T. Warburton, D. Gottlieb, High-order/spectral unstructured grid methods for conservation laws, ICOSAHOM-01 International Conference on Spectral and High Order Methods, June 11–15, Book of Abstract, Uppsala University, Sweden, 2001.
[10] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, Springer, Berlin, Heidelberg, 1994.
[11] T. Granlund, The GNU Multiple Precision Arithmetic Library Edition 4.1.3, 2004. http://www.swox.com/gmp.