1 Classification of binary cubic self-dual codes

1. $\ell = 2$
It is easy to show that there is a unique self-dual code C_2 over R up to equivalence of Definition 2.2. The generator matrix is

$$G_2 = \left[\begin{array}{cc} 1 & 1 \end{array}\right].$$

The corresponding binary cubic self-dual code is a decomposable code, denoted by $C_2 \oplus C_2 \oplus C_2$ in the notation of [3, Table 2].

2. $\ell = 4$
It is easy to show that there are exactly two inequivalent quasi-cyclic self-dual codes, $C_{4,1}, C_{4,2}$ in R. The generator matrix of $C_{4,1}$ is

$$G_{4,1} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \end{array}\right].$$

The corresponding binary cubic self-dual code is C_2^6 in the notation of [3, Table 2].
The generator matrix of $C_{4,2}$ is

$$G_{4,2} = \left[\begin{array}{cc|cc} 1 & 0 & Y+1 & Y^2+Y+1 \\ \hline Y & Y & 1 & 1 \end{array}\right].$$

The corresponding binary cubic self-dual code is B_{12} in the notation of [3, Table 2]. We note that $\phi^{-1}(C_{4,2})$ is the unique $[12, 6, 4]$ extremal Type I code.

3. $\ell = 6$
There are 1920 self-dual codes over R of length 6 constructed from $G_{4,1}$ and $G_{4,2}$ by using Theorem 2.3. We can verify that there are exactly three permutation inequivalent binary cubic self-dual codes, $\phi^{-1}(C_{6,1}), \phi^{-1}(C_{6,2}), \phi^{-1}(C_{6,3})$. The generator matrix of $C_{6,1}$ is

$$G_{6,1} = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

The corresponding binary cubic self-dual code is C_2^9 in the notation of [3, Table 2].
The generator matrix $G_{6,2}$ of $C_{6,2}$ is

$$\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & Y+1 & Y^2+Y+1 \\ \hline Y^2+Y+1 & Y^2+Y+1 & 1 & 0 & 0 & 1 \\ Y & Y & 1 & 1 & 1 & 1 \end{array}\right].$$

The corresponding binary cubic self-dual code is H_{18} in the notation of [3, Table 2]. We note that $\phi^{-1}(C_{6,2})$ is a $[18, 9, 4]$ extremal Type I code.
The generator matrix of $C_{6,3}$ is

$$G_{6,3} = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & Y+1 & Y^2+Y+1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ Y & Y & 1 & 1 & 1 & 1 \end{array}\right].$$

The corresponding binary cubic self-dual code is $C_2^3 \oplus B_{12}$ in the notation of [3, Table 2].

4. $\ell = 8$
We construct all self-dual codes over R of length 8 from 1920 self-dual codes over R of length 6 by using the method in Theorem 2.3. We can verify that there are exactly

Table 1: All binary cubic self-dual codes of length 24

binary codes	sd codes (PleSlo75)	min. wt d
$\phi^{-1}(C_{8,1})$	decomposable	2
$\phi^{-1}(C_{8,2})$	decomposable	2
$\phi^{-1}(C_{8,3})$	decomposable	2
$\phi^{-1}(C_{8,4})$	decomposable	4
$\phi^{-1}(C_{8,5})$	V_{24}	4
$\phi^{-1}(C_{8,6})$	A_{24}	4
$\phi^{-1}(C_{8,7})$	D_{24}	4
$\phi^{-1}(C_{8,8})$	C_{24}	4
$\phi^{-1}(C_{8,9})$	W_{24}	4
$\phi^{-1}(C_{8,10})$	decomposable	4
$\phi^{-1}(C_{8,11})$	Z_{24} (odd Golay)	6
$\phi^{-1}(C_{8,12})$	Q_{24}	4
$\phi^{-1}(C_{8,13})$	F_{24}	4
$\phi^{-1}(C_{8,14})$	G_{24} (Golay)	8
$\phi^{-1}(C_{8,15})$	L_{24}	4
$\phi^{-1}(C_{8,16})$	E_{24}	4

16 binary cubic self-dual codes up to permutation equivalence, and we denote them by $\phi^{-1}(C_{8,1}), \ldots, \phi^{-1}(C_{8,16})$. We construct $C_{8,1}, \ldots, C_{8,9}$ from $G_{6,1}$, $C_{8,10}, \ldots, C_{8,14}$ from $G_{6,2}$, and $C_{8,15}$ and $C_{8,16}$ from $G_{6,3}$. For the generator matrices of $C_{8,1}, \cdots, C_{8,16}$, we only list $\mathbf{x}$ vectors in the notation of Theorem 2.3, respectively:

$$
\begin{aligned}
&C_{8,1} : \mathbf{x} = (0,0,0,0,0,1)\\
&C_{8,2} : \mathbf{x} = (0,0,0,0,Y+1,Y^2+Y+1)\\
&C_{8,3} : \mathbf{x} = (0,0,0,Y+1,Y^2+Y+1,0)\\
&C_{8,4} : \mathbf{x} = (0,0,1,0,1,1)\\
&C_{8,5} : \mathbf{x} = (0,0,1,0,Y+1,Y+1)\\
&C_{8,6} : \mathbf{x} = (0,0,1,0,Y^2+Y+1,Y^2+Y+1)\\
&C_{8,7} : \mathbf{x} = (0,0,1,1,Y+1,Y^2+Y+1)\\
&C_{8,8} : \mathbf{x} = (0,0,1,Y+1,Y^2+Y+1,1)\\
&C_{8,9} : \mathbf{x} = (0,0,Y+1,1,Y,Y^2+Y+1)\\
&C_{8,10} : \mathbf{x} = (0,0,0,0,Y^2+Y+1,Y+1)\\
&C_{8,11} : \mathbf{x} = (0,0,0,1,Y+1,Y+1)\\
&C_{8,12} : \mathbf{x} = (0,0,0,1,Y^2+Y+1,Y^2+Y+1)\\
&C_{8,13} : \mathbf{x} = (0,0,1,1,Y+1,Y^2+Y+1)\\
&C_{8,14} : \mathbf{x} = (0,0,1,Y+1,1,Y^2+Y+1)\\
&C_{8,15} : \mathbf{x} = (0,0,Y+1,Y+1,Y+1,Y^2+Y+1)\\
&C_{8,16} : \mathbf{x} = (0,0,Y+1,Y^2+Y+1,Y^2+Y+1,Y^2+Y+1)
\end{aligned}
$$

5. $\ell = 10$, $[30, 15, 6]$ codes

There are three weight enumerators for self-dual $[30, 15, 6]$ codes [2]:

$W_1 = 1 + 19y^6 + 393y^8 + 1848y^{10} + 5192y^{12} + \cdots$.

$W_2 = 1 + 27y^6 + 369y^8 + 1848y^{10} + 5256y^{12} + \cdots$.

$W_3 = 1 + 35y^6 + 345y^8 + 1848y^{10} + 5320y^{12} + \cdots$.

It is known [1], [2] that there are precisely three codes with W_1, a unique code with W_2, and precisely nine codes with W_3. Only two cubic self-dual $[30, 15, 6]$ codes are given in http://kutacc.kut.ac.kr/˜sunghyu/data/qcsd.htm. We have constructed three codes with W_1 whose group orders are $576, 1152, 18432$ respectively. We have also constructed five codes with W_3 whose group orders are $30, 192, 1440, 40320, 645120$. These codes are as follows:

G_1:

$[1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ Y+1\ Y^2+Y+1]$

$[Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 0\ 0\ 1\ Y+1\ Y+1]$

$[1\ 1\ Y^2+Y\ Y^2+Y\ 1\ 0\ 0\ 0\ Y+1\ Y^2+Y+1]$

$[Y^2+Y\ Y^2+Y\ Y^2+1\ Y^2+1\ Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 1]$

$[Y\ Y\ 1\ 1\ Y\ Y\ 1\ 1\ 1\ 1]$

C_1 : *Linear code* $[30, 15, 6]$ *with* W_1

[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 0 1 1 0 1 1]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 1 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 0 0 0 1 1 0 0 1 1]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 0 0 1 0 1 1 1 1]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 1 1 0 1 0 0 0 1 1]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 1 0 0 1 1 0 0 0 1]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 1 0 0 1 1 1 0 1 0 0 0 0 1]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 1 0 1 0 0 1 0 1 1 0 1 0 0]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 1 0 1 1 1 1 0 1 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 1 1 0 1 1 1 0 1 1 0 0 1]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 0 0 0 1 1 1 0 1 1 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 0 1 0 0 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 0 1 0 1 0 0 0 1 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0 1 1 0 1 1 1 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 1 0 1 1 0 1 0 0 1 1]
Automorphism group order : 576

G_2:

$[1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ Y^2+Y+1\ Y^2+1]$

$[Y^2\ Y^2\ 1\ 0\ 0\ 0\ 0\ 1\ Y+1\ Y+1]$

$[0\ 0\ Y^2+Y\ Y^2+Y\ 1\ 0\ 0\ 0\ Y+1\ Y^2+Y+1]$

$[Y\ Y\ Y^2+1\ Y^2+1\ Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 1]$

$[Y^2\ Y^2\ 1\ 1\ Y\ Y\ 1\ 1\ 1\ 1]$

C_2 : *Linear code* $[30, 15, 6]$ *with* W_1

[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 0 0 0 0 1 0 1 0 1]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 1 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1 1 1 0 1 0 1 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1 1 0 1 0 0 1 1 1]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1 0 1 0 1 0 1 1 1 1]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 1 0 0 1 1 1 1 0 1]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 1 0 0 1 1 1 1 1 1 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 1 1 0 0 0 0]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 1 1 0 1 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 1 1 1 1 0 1 1 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 1 0 0 1 0 0 0 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1 0 1 1 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 1 1 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 0 1 0 1 1 0 1 1]
Automorphism group order : 1152

G_3:
$[1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ Y^2+Y\ Y^2+Y+1]$
$[Y\ Y\ 1\ 0\ 0\ 0\ 0\ 1\ Y+1\ Y+1]$
$[Y^2\ Y^2\ Y^2+Y\ Y^2+Y\ 1\ 0\ 0\ 0\ Y+1\ Y^2+Y+1]$
$[0\ 0\ Y^2+1\ Y^2+1\ Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 1]$
$[Y\ Y\ 1\ 1\ Y\ Y\ 1\ 1\ 1\ 1]$

C_3 : *Linear code* $[30, 15, 6]$ *with* W_1
[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 1 1 1 0 0 1 1 0 1]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 0 1 0 1 1 1 1 1]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 1 0 1 0 0 0 0 1]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 1 1 0 1 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 1 1 1 0 0 1 0 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 0 1 1 1 0 0 1]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 1 0 0 1 1 1 1 1 1 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 1 0 0 1 1 1 1 0 1 1]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 1 0 0 1 0 0 0 1 1 0 0 1]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 1 0 1 1 1 0 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 1 0 1 0 0 0 0 1 0 1 1 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 1 1 1 1 0 1 1 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 1 1 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 1 1 0 0 1 0 1 0 0]
Automorphism group order : 18432

G_4: $[1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ Y+1\ Y^2+1]$
$[Y\ Y\ 1\ 0\ 0\ 0\ 0\ 1\ Y+1\ Y+1]$
$[Y^2+Y\ Y^2+Y\ Y^2+Y\ Y^2+Y\ 1\ 0\ 0\ 0\ Y+1\ Y^2+Y+1]$

$[Y+1\ Y+1\ Y^2+1\ Y^2+1\ Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 1]$
$[Y^2+Y+1\ Y^2+Y+1\ 1\ 1\ Y\ Y\ 1\ 1\ 1\ 1]$

C_4 : *Linear code* $[30, 15, 6]$ *with* W_3
[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 1 1 0 0 0 1 1]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 1 1 0 0 1 1 0 1 0 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 1 1 1 1 1 1]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1 1 0 1 1 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 1 1 1 0 0 0 1 1 0 0 1 1]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 1 1 0 0 1 1 1 1 1 1 1 0 1 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 0 1 0 1 1 1 0 1 1 0 0 1 0]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 1]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 1 0 1 0 0 0 1 0 1 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 0 1 0 0 0 0 1 0 1 0 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 1 0 1 1 1 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1 0 1 1 0 1 1 1 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 1 1 1 0 1 0 0 0 1]
Automorphism group order : 192

G_5: $[1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ Y^2+1\ Y^2+Y]$
$[Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 0\ 0\ 1\ Y+1\ Y+1]$
$[Y^2+1\ Y^2+1\ Y^2+Y\ Y^2+Y\ 1\ 0\ 0\ 0\ Y+1\ Y^2+Y+1]$
$[Y^2+Y\ Y^2+Y\ Y^2+1\ Y^2+1\ Y^2+Y+1\ Y^2+Y+1\ 1\ 0\ 0\ 1]$
$[Y^2\ Y^2\ 1\ 1\ Y\ Y\ 1\ 1\ 1\ 1]$

C_5 : *Linear code* $[30, 15, 6]$ *with* W_3
[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 1 0 0]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 1 0 0 1 1 0 1 1 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 1 0 1 1 0 1 1 1 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1 0 0 0 1 0 0 0 0 1]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 1 1 0 1 1 0 0 1 1]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 1 0 1 0 1 1]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0 1 1 1 1 1 0 1 0 0 1]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 1 0 1 1 1 1 0 1 0 1 1 0]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 1 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 1 1 0 1 1 1 0 1 0 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 1 0 1 0 0 0 1 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 1 0 0 1 0 1 0 1 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 1 1 0 0 0 1 1 0 1 1 1 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0 0 0 0 0 0 0 1 1 0]
Automorphism group order : 1440

G_6: [1 0 0 0 0 0 0 $Y+1$ 1 Y^2+Y]
[Y^2+1 Y^2+1 1 0 0 0 0 1 $Y+1$ $Y+1$]
[$Y+1$ $Y+1$ Y^2+Y Y^2+Y 1 0 0 0 $Y+1$ Y^2+Y+1]
[Y^2+Y Y^2+Y Y^2+1 Y^2+1 Y^2+Y+1 Y^2+Y+1 1 0 0 1]
[Y Y 1 1 Y Y 1 1 1 1]

C_6 : *Linear code* [30, 15, 6] *with* W_3
[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 1 1 0 0 0 1 1]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 1 1 1 1 1 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 1 1 0 1 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 1 1 1 1 1 0 1 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 0 0 1 0 1 0 1 1]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 0 1 1 1 0 0 1]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 1 0 1 0 0 0 0 1]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0 1 1 1 1 1 0 1 0 0 1]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 0 1 0 0 1 0 0 0 1 0 1 1]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 1 1 0 1 1 1 1 0 1 1 1 1]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 0 1 0 1 1 1 1 0 1 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 1 1 1 0 0 0 0 1 0 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 0 1 0 1 0 1 1 1 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 0 1 1 0 0 0 1 1 0 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 0 1 1 1 1 0 0 1 0 1]
Automorphism group order : 30

G_7: [1 0 0 0 0 1 1 1 Y^2+Y Y^2+Y]
[1 1 1 0 0 0 0 1 $Y+1$ $Y+1$]
[$Y+1$ $Y+1$ Y^2+Y Y^2+Y 1 0 0 0 $Y+1$ Y^2+Y+1]
[0 0 Y^2+1 Y^2+1 Y^2+Y+1 Y^2+Y+1 1 0 0 1]
[Y Y 1 1 Y Y 1 1 1 1]

C_7 : *Linear code* [30, 15, 6] *with* W_3
[1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 0 0 1 1 0 1 0 0]
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 1 1 1 1 0 0 1 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 1 0 1 1 0 1 1 1 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1 1 0 1 0 0 1 1 1]
[0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1]
[0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 1 1 0 0 0 1 0 1 0]
[0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 1 1 0 0 1 1 1 0 0 1 0 0 1 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 1 0 1 1 1 1 0 1 1 0 1 0 0]
[0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 1 0 0 1 1 0 0 1 1 1 1 1 1 0]
[0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 0 0 1 0 1 1 1 1 1 1 0 0 1]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 0 0 1 1 1 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0 0 0 0 1 0 1 1 0 1 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 0 1 0 1 0 1 0 1 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 0 1 0 1 1 0 1 1]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 0 1 1 0 1 1]

Table 2: Binary extremal cubic self-dual codes of length $n = 36, 42$

Codes $C_{n,i}$	X vector	Using Gen. Matrix	Weight Enumerator	|Aut|
$C_{36,1}$	$(Y^2+Y+1, Y^2+1, Y^2+Y, Y+1, Y^2+Y,$ $Y^2+1, Y, Y^2+1, Y^2+Y, Y^2)$	G_{10}	W_1	18
$C_{36,2}$	$(1, Y^2+Y+1, Y+1, 1, Y^2+Y,$ $Y+1, Y^2+Y, 1, Y, Y+1)$	G_{10}	W_1	24
$C_{36,3}$	$(Y^2+1, 1, Y^2+Y+1, Y+1, Y^2+Y+1,$ $Y, Y^2+1, 0, Y^2+1, Y)$	G_{10}	W_1	36
$C_{36,4}$	$(Y^2+1, Y+1, 1, Y+1, 0,$ $Y, Y+1, 0, 0, 1)$	G_{10}	W_1	48
$C_{36,5}$	$(Y^2, 1, Y^2, Y+1, 0,$ $Y^2+Y, Y^2+1, Y^2+Y, Y^2, Y)$	G_{10}	W_1	96
$C_{36,6}$	$(Y, 0, Y^2+1, Y+1, Y^2+1,$ $Y+1, 1, Y^2+Y+1, 0, Y+1)$	G_{10}	W_1	240
$C_{36,7}$	$(Y^2+Y, Y^2+Y, Y^2+Y, Y^2+Y, Y^2+Y,$ $Y, 1, Y^2, Y^2+Y+1, Y^2)$	G_{10}	W_1	288
$C_{36,8}$	$(Y^2+1, Y^2+1, Y^2+Y+1, Y^2+Y, 1,$ $1, Y^2+Y, Y+1, Y^2, Y^2)$	G_{10}	W_1	384
$C_{36,9}$	$(Y^2+1, 0, Y, 0, 1,$ $Y^2, 0, Y^2, 1, Y+1)$	G_{10}	W_1	12960
$C_{42,1}$	$(Y^2+Y, Y+1, 1, Y, Y^2+Y+1, Y,$ $Y, 0, Y^2+Y, Y^2, 10)$	G_{12}	$W_2,\ \beta=0$	3
$C_{42,2}$	$(Y^2+Y, Y^2+1, Y, Y^2+1, Y^2+1, Y+1,$ $Y^2+Y, 0, 1, Y, Y^2+Y, Y^2+1)$	G_{12}	$W_2,\ \beta=3$	3
$C_{42,3}$	$(Y, Y^2, 1, Y^2+Y, 1, Y, Y, Y^2+Y+1,$ $Y^2+Y, Y^2+Y+1, 0, Y)$	G_{12}	$W_2,\ \beta=6$	3
$C_{42,4}$	$(Y^2+1, Y+1, 0, Y^2+Y+1, Y^2+Y,$ $Y^2, Y^2+Y, Y^2+1, Y^2+Y, Y^2, Y^2+1, 0)$	G_{12}	$W_2,\ \beta=9$	3
$C_{42,5}$	$(Y^2, 1, Y, Y^2, Y, 0, Y^2, Y^2,$ $Y^2, Y^2+Y, Y^2+Y+1, 0)$	G_{12}	$W_2,\ \beta=0$	6
$C_{42,6}$	$(0, Y^2, Y^2+Y+1, 0, Y^2+Y, 1, 0, 1,$ $Y^2+Y+1, Y, Y^2+Y+1, 0)$	G_{12}	$W_2,\ \beta=3$	6
$C_{42,7}$	$(Y^2+Y+1, Y^2+Y, Y^2+Y, Y^2+Y, Y+1,$ $Y^2+1, Y+1, Y, Y^2+1, 0, Y^2, 0)$	G_{12}	$W_2,\ \beta=6$	6
$C_{42,8}$	$(Y^2+Y+1, 1, Y, Y+1, Y+1, Y^2+Y,$ $Y^2, Y, Y^2+Y, Y^2+1, Y, Y)$	G_{12}	$W_2,\ \beta=9$	6
$C_{42,9}$	$(Y^2, Y, 0, Y^2+Y+1, Y^2, Y^2+1, Y^2+Y+1,$ $1, Y, Y, Y^2+Y+1, 0)$	G_{12}	$W_2,\ \beta=12$	6
$C_{42,10}$	$(Y, 1, 0, Y^2+Y+1, Y^2, Y^2+Y+1, 1,$ $Y^2+Y, Y^2+Y+1, Y^2+1, Y^2+Y+1, 1)$	G_{12}	$W_2,\ \beta=0$	12
$C_{42,11}$	$(0, Y, Y^2+Y+1, Y^2+Y+1, Y^2+Y,$ $Y^2+1, Y^2+Y+1, 0, 0, Y^2+Y, 0, Y)$	G_{12}	$W_2,\ \beta=3$	12
$C_{42,12}$	$(1, Y^2+Y+1, 1, 1, Y^2, 1, Y^2+1,$ $Y^2+Y, Y^2+Y+1, 0, Y, 1)$	G_{12}	$W_2,\ \beta=9$	12
$C_{42,13}$	$(Y^2+1, 0, 1, Y^2, 1, Y, Y+1, Y^2,$ $Y^2+1, Y^2+Y, Y, Y^2)$	G_{12}	$W_2,\ \beta=6$	36
$C_{42,14}$	$(1, Y^2+1, Y^2+1, 0, Y^2+1, Y^2+1,$ $Y+1, Y^2, Y+1, Y, Y^2+Y, Y+1)$	G_{12}	$W_2,\ \beta=9$	36

Automorphism group order : 40320

6. $\ell = 11, 12$, $[36, 18, 8]$, $[42, 21, 8]$ codes

$$G_{10} = \begin{pmatrix} 1, 0, Y+1, Y, 1, 1, Y^2+Y, Y^2+1, Y^2+Y, 0 \\ 1, 1, 1, 0, Y^2, Y^2+Y, Y^2, Y^2+Y+1, Y^2+1, Y^2+Y \\ Y^2+1, Y^2+1, Y^2+Y+1, Y^2+Y+1, 1, 0, Y+1, Y^2+1, Y^2+Y, Y^2+Y+1 \\ Y^2+Y, Y^2+Y, Y+1, Y+1, 1, 1, 1, 0, Y+1, Y^2+Y+1 \\ 1, 1, Y^2, Y^2, Y^2, Y^2, Y, Y, 1, 1 \end{pmatrix}$$

$$G_{12} = \begin{pmatrix} 1, 0, Y^2, Y^2+Y+1, Y^2+Y, Y^2+Y+1, Y^2+Y+1, Y^2+1, Y^2, Y+1, Y, Y \\ 0, 0, 1, 0, 0, Y^2+Y, Y, Y^2, Y^2+1, Y, Y^2+Y+1, Y^2+Y+1 \\ Y^2+1, Y^2+1, Y^2+1, Y^2+1, 1, 0, Y^2, Y^2+Y, Y^2, Y^2+Y+1, Y^2+1, Y^2+Y \\ 1, 1, 0, 0, Y^2+Y+1, Y^2+Y+1, 1, 0, Y+1, Y^2+1, Y^2+Y, Y^2+Y+1 \\ Y, Y, 1, 1, Y+1, Y+1, 1, 1, 1, 0, Y+1, Y^2+Y+1 \\ 1, 1, Y^2+Y+1, Y^2+Y+1, Y^2, Y^2, Y^2, Y^2, Y, Y, 1, 1 \end{pmatrix}$$

2 Classification of binary quintic self-dual codes

1. Case: $\ell = 2$
   It can be checked that there is a unique self-dual code C_2 in R. The generator matrix is
   $$G_2 = \left[\begin{array}{cc} 1 & 1 \end{array}\right].$$
   The weight enumerator of the corresponding quintic self-dual code is
   $$W_{\phi^{-1}(C_2)} := (1+y^2)^5.$$

2. Case: $\ell = 4$
   There are 120 self-dual codes over R of length 4 constructed from G_2 by using Theorem 2.3. We can verify that there are exactly three permutation inequivalent binary quintic self-dual codes, $\phi^{-1}(C_{4,1}), \phi^{-1}(C_{4,2}), \phi^{-1}(C_{4,3})$, and in fact, they are all the possible binary quintic self-dual codes of length 20 by Theorem 2.7.

   The generator matrix of $C_{4,1}$ is
   $$G_{4,1} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \end{array}\right].$$
   The quintic self-dual code corresponding to $G_{4,1}$ is equivalent to C_2^{10} in [3].

   The generator matrix of $C_{4,2}$ is
   $$G_{4,2} = \left[\begin{array}{cc|cc} 1 & 0 & Y+1 & Y^3+Y+1 \\ \hline Y^2 & Y^2 & 1 & 1 \end{array}\right].$$
   The quintic self-dual code corresponding to $G_{4,2}$ has the following weight enumerator
   $$W_{\phi^{-1}(C_{4,2})} := 1 + 5\,y^4 + 80\,y^6 + 250\,y^8 + 352y^{10} + \cdots,$$
   and the order of automorphism group is $2^{13} \cdot 3 \cdot 5$. The code $\phi^{-1}(C_{4,2})$ is equivalent to M_{20} in [3].

   The generator matrix of $C_{4,3}$ is
   $$G_{4,3} = \left[\begin{array}{cc|cc} 1 & 0 & Y^3+Y^2+Y+1 & Y^4+Y^3+Y^2+Y+1 \\ \hline Y & Y & 1 & 1 \end{array}\right].$$
   The quintic self-dual code corresponding to $G_{4,3}$ has the following weight enumerator
   $$W_{\phi^{-1}(C_{4,2})} := 1 + 45\,y^4 + 210\,y^8 + 512y^{10} + \cdots,$$
   and the order of automorphism group is $2^{17} \cdot 3^4 \cdot 5^2 \cdot 7$. The code $\phi^{-1}(C_{4,3})$ is of Type II and equivalent to J_{20} in [3].

3. Case: $\ell = 6$
   We construct all self-dual codes over R of length 6 from 120 self-dual codes over R of length 4 by using the building-up construction in Theorem 2.3. We can verify that there are exactly 11 quintic self-dual codes up to permutation equivalence, and we denote them by $\phi^{-1}(C_{6,1}), \ldots, \phi^{-1}(C_{6,11})$; Theorem 2.7 shows that they are all the possible

binary quintic self-dual codes of length 30. We construct $C_{6,1},\ldots,C_{6,10}$ from $G_{4,1}$, and $C_{6,11}$ from $G_{4,2}$. The generator matrices are given as follows:

$$G_{6,1}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,2}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & Y+1 & Y^3+Y+1 \\ \hline Y^4+Y^2+1 & Y^4+Y^2+1 & 1 & 0 & 0 & 1 \\ Y^2 & Y^2 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,3}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & Y^3+Y^2+Y+1 & Y^4+Y^3+Y^2+Y+1 \\ \hline Y^4+Y^3+Y^2+Y+1 & Y^4+Y^3+Y^2+Y+1 & 1 & 0 & 0 & 1 \\ Y & Y & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,4}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & Y & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \\ Y^4 & Y^4 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,5}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & Y^2+Y & Y+1 \\ \hline Y^4+1 & Y^4+1 & 1 & 0 & 0 & 1 \\ Y^3 & Y^3 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,6}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & Y^3+Y^2 & Y+1 \\ \hline Y^4+1 & Y^4+1 & 1 & 0 & 0 & 1 \\ Y^4+Y^3+Y^2 & Y^4+Y^3+Y^2 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,7}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & Y+1 & Y^2+1 & Y^4+Y^3+Y^2+Y+1 \\ \hline Y^4+Y^3+Y^2+Y+1 & Y^4+Y^3+Y^2+Y+1 & 1 & 0 & 0 & 1 \\ Y^2+Y+1 & Y^2+Y+1 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,8}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & Y+1 & Y^2+Y & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 1 \\ Y^3 & Y^3 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,9}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & Y+1 & Y^3+Y+1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ Y^2 & Y^2 & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,10}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & Y^3+Y^2+Y+1 & Y^4+Y^3+Y^2+Y+1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ Y & Y & 1 & 1 & 1 & 1 \end{array}\right].$$

$$G_{6,11}=\left[\begin{array}{cc|cccc} 1 & 0 & 0 & Y+1 & Y+1 & 1 \\ \hline Y^4+Y^3+1 & Y^4+Y^3+1 & 1 & 0 & Y+1 & Y^3+Y+1 \\ Y^4+Y^2+Y & Y^4+Y^2+Y & Y^2 & Y^2 & 1 & 1 \end{array}\right].$$

The corresponding weight enumerators of the quintic self-dual codes $\phi^{-1}(C_{6,i})$ with $i=1,2,\cdots,11$ are given as follows. In fact, $W_{\phi^{-1}(C_{6,2})}$, $W_{\phi^{-1}(C_{6,5})}$, $W_{\phi^{-1}(C_{6,6})}$, and $W_{\phi^{-1}(C_{6,11})}$ are the same, and so there are exactly 8 different weight enumerators as follows:

$$\begin{aligned}
W_{\phi^{-1}(C_{6,1})} &= 1+15y^2+105y^4+455y^6+1365y^8+3003y^{10}+5005y^{12}+6435y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,2})} &= W_{\phi^{-1}(C_{6,5})}=W_{\phi^{-1}(C_{6,6})}=W_{\phi^{-1}(C_{6,11})}\\
&= 1+35y^6+345y^8+1848y^{10}+5320y^{12}+8835y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,3})} &= 1+30y^4+125y^6+315y^8+1518y^{10}+5140y^{12}+9255y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,4})} &= 1+15y^4+80y^6+330y^8+1683y^{10}+5230y^{12}+9045y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,7})} &= 1+10y^4+65y^6+335y^8+1738y^{10}+5260y^{12}+8975y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,8})} &= 1+5y^4+50y^6+340y^8+1793y^{10}+5290y^{12}+8905y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,9})} &= 1+5y^2+15y^4+115y^6+705y^8+2453y^{10}+5335y^{12}+7755y^{14}+\cdots.\\
W_{\phi^{-1}(C_{6,10})} &= 1+5y^2+55y^4+235y^6+665y^8+2013y^{10}+5095y^{12}+8315y^{14}+\cdots.
\end{aligned}$$

The orders of the automorphism groups of $\phi^{-1}(C_{6,1}), \ldots, \phi^{-1}(C_{6,11})$ are as follows respectively: $2^{26} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, $2^5 \cdot 3^2 \cdot 5$, $2^{22} \cdot 3^4 \cdot 5^3$, $2^{16} \cdot 3^5 \cdot 5$, $2 \cdot 3 \cdot 5$, $2^7 \cdot 3^2 \cdot 5 \cdot 7$, $2^{15} \cdot 3^2 \cdot 5^2$, $2^{11} \cdot 5$, $2^{21} \cdot 3^2 \cdot 5^2$, $2^{25} \cdot 3^5 \cdot 5^3 \cdot 7$, $2^{11} \cdot 3^2 \cdot 5 \cdot 7$.

References

[1] J.H. Conway, V. Pless, N.J.A. Solane, The binary self-dual codes of length up to 32, a revised enumeration. J. Combin. Theory, Ser. A. 60 (1992) 183-195.

[2] J.H. Conway, N.J.A. Solane, A new upper bound on the minimal distance of self-dual codes, IEEE Trans. Inform. Theory. 36 (1990) 1319-1333.

[3] Pless V.: A classification of self-orthogonal codes over $GF(2)$. Discrete Math. 3, 209-246 (1972).